Categorical Logic
Exercise Sheet 1

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An initial object 0 in a category is *strict* if, for every object X, every morphism $X \to 0$ is an isomorphism.

1. To our language with products, add a type 0 and an expression former $\mathsf{absurd}(e)$. There is a typing rule
$$\frac{\Gamma \vdash e : 0}{\Gamma \vdash \mathsf{absurd}(e) : T}$$
and an equation
$$\frac{\Gamma \vdash e : 0 \qquad \Gamma \vdash e' : T}{\Gamma \vdash \mathsf{absurd}(e) = e' : T}$$
for every type T and expression e.

   Define a semantics of this language in a category with finite products and an initial object. Is the semantics sound? i.e., if $\Gamma \vdash e = e' : T$ is derivable, can you conclude that the morphism $[\![\Gamma \vdash e : T]\!]$ is equal to the morphism $[\![\Gamma \vdash e' : T]\!]$?

2. Complete the Agda examples at `http://www.cl.cam.ac.uk/teaching/0910/L20/`

3. On strict initial objects:

   (a) Which of the following categories are have strict initial objects?
      i. The one object category;
      ii. The category $\Sigma = (\bullet \to \bullet)$.
      iii. The category of sets;
      iv. The category of arrows;

   (b) For any set X, we write $\mathrm{Sub}(X)$ for the poset of subsets of X ordered by inclusion. Considered as a category, does $\mathrm{Sub}(X)$ have a strict initial object? Is it cartesian closed?

   (c) Let $\mathcal{C}$ and $\mathcal{D}$ be categories, and consider a functor $F: \mathcal{C} \to \mathcal{D}$. Prove that if F has a right adjoint and 0 is an initial object of $\mathcal{C}$ then $F(0)$ is an initial object of $\mathcal{D}$.

   (d) Consider a category with binary products and an initial object. Show that if the projection $X \times 0 \to 0$ is an isomorphism for every object X, then 0 is a strict initial object. You could use the internal language for products.

   (e) Prove that an initial object in a cartesian closed category is strict.

   (f) A *pointed set* is a set X together with an element $x \in X$. We write (X, x) for a pointed set. A *pointed function* $(X, x) \to (Y, y)$ is a function $f: X \to Y$ such that $f(x) = y$. Pointed sets and pointed functions form a category. Prove that it has finite products and an initial object, but that it is not cartesian closed.

   (g) Prove that the category of monoids is not cartesian closed.