Categorical Logic
Exercise Sheet 3

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Images, covers, and regular categories.

1. Consider the monoids of natural numbers under addition $(\mathbb{N}, +, 0)$ and of integers under addition $(\mathbb{Z}, +, 0)$. Show that the inclusion $\mathbb{N} \to \mathbb{Z}$ is a monoid homomorphism. Show that it is a monomorphism and an epimorphism. Is it a cover?

2. Consider a morphism $f : A \to B$ in a category with finite limits and images. Explain what is meant by the sequent $\top \vdash_y \exists x.\ f(x) = y$.

3. In a category with pullbacks, for every morphism $f : A \to B$ there is a monotone function $f^* : \mathrm{Sub}(B) \to \mathrm{Sub}(A)$, taking a subobject of B to its inverse image along f.

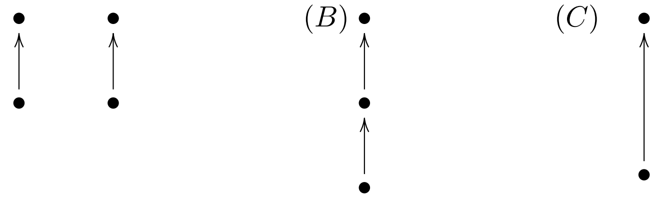
   In a category with images, for every morphism $f : A \to B$ there is a monotone function $\exists_f : \mathrm{Sub}(A) \to \mathrm{Sub}(B)$, taking a subobject $S \rightarrowtail A$ to the image of the composite $(S \rightarrowtail A \xrightarrow{f} B)$.

   Show that in a category with pullbacks and images, f^* is right adjoint to $\exists_f$. (In fact, a category has images if and only if f^* has a left adjoint for all f.)

4. Show that the category of sets is regular.

5. Consider the following posets:

   (A) (B) (C)



   (a) What is a monomorphism in the category of posets and monotone maps?
   (b) Describe a cover $f : A \to B$.
   (c) Consider the map $g : C \to B$ that preserves top and bottom elements. What is the pullback of f along g? Explain why the category of posets is not regular.

6. We now prove that the category of monoids is regular.

   (a) Show that the forgetful functor from the category of monoids to the category of sets preserves pullbacks.
   (b) Consider a monoid homomorphism $f : A \to B$. Show that the direct image of f, i.e. the set $\{b \in B \mid \exists a \in A.\ f(a) = b\}$, forms a submonoid of B. Show that it is the image of f in the category of monoids. Explain why the forgetful functor from the category of monoids to the category of sets preserves covers.
   (c) Using (4) and (a) and (b), conclude that the category of monoids is regular.