Homework Assignment for Randomized Algorithms

QUESTIONS

1) (Chernoff Bound)
   a) Prove Chernoff bound:

$$\mathbb{P}\{X \geq x\} \leq \min_{t>0} \frac{\mathbb{E}[e^{tX}]}{e^{tx}} \tag{1}$$

   b) Apply Chernoff bound to Poisson random variable X, and show:

$$\mathbb{P}\{X \geq x\} \leq \frac{e^{-\mu}(e\mu)^x}{x^x}, \quad \text{if } x > \mu \tag{2}$$

$$\mathbb{P}\{X \leq x\} \leq \frac{e^{-\mu}(e\mu)^x}{x^x}, \quad \text{if } x < \mu \tag{3}$$

   where μ is the mean of X.
2) (Balls and Bins) There are n balls thrown independently and uniformly at random into n bins.
   a) Show the probability that a particular bin receiving at least M balls is at most $\binom{n}{M}(\frac{1}{n})^M$.
   b) Show the probability that any bin receiving more than $\frac{3\ln n}{\ln\ln n}$ balls is at most $\frac{1}{n}$.
   c) Derive a sufficient number of n that can guarantee that the probability any bin receiving more than 1% of balls is at most 1%.
3) (Bit Strings) An ideal hash function will map a string to an bit-string, such that each bit has a equal probability of being 0 or 1. Suppose that there are a set of strings $S = \{s_1, ..., s_m\}$. We map each string to a fingerprint bit-string using b bits by an ideal hashing function. Then we store the set of fingerprints $F = \{f_1, ..., f_m\}$, which are used to validate whether a new string is a member of S or not.
   a) Show that $b = \Omega(\log m)$ bits is necessary for the probability of a false positive being lesser than 1.
   b) Show that $b = O(\log m)$ bits is sufficient for the probability of a false positive being at most $\frac{1}{m}$.