

# *Computer Systems Modelling*



Computer Laboratory

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Computer Science Tripos, Part II

Lent Term 2010/11

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Problem sheet

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### Examples (i)

1. In what contexts is simulation an appropriate technique for performance evaluation? When it is inappropriate?
2. How would you generate random variates for the exponential distribution? Write pseudo code for a simulation component which models the arrival process of customers at a bank, given that the mean arrival rate is 40 customers per hour.
3. How would you generate pseudo random variables from the mixture of exponential distributions with probability density function

$$f(x) = 1/3\lambda_1 e^{-\lambda_1 x} + 2/3\lambda_2 e^{-\lambda_2 x}$$

where  $\lambda_1 = 5$  and  $\lambda_2 = 10$ ?

4. Suppose that a simulation is constructed to estimate the mean response time, in milliseconds, of an interactive computer system. A number of repetitions are performed with the following results (measured in milliseconds): 4.1, 3.6, 3.1, 4.5, 3.8, 2.9, 3.4, 3.3, 2.8, 4.5, 4.9, 5.3, 1.9, 3.7, 3.2, 4.1, 5.1.

Calculate the sample mean and variance of these results and hence derive a 95% confidence interval for the mean response time. You may wish to make use of the fact that  $\mathbb{P}(Z > 1.96) > 0.025 > \mathbb{P}(Z > 1.97)$  where  $Z$  is a random variable with the standard normal distribution,  $N(0, 1)$ .

5. An experiment is performed to estimate the performance of a  $M/M/1$  system with FIFO queueing. The response times of each of 1000 successive customers are recorded and the sample mean ( $\bar{X}$ ) and sample variance ( $S^2$ ) of these numbers are calculated.

A student believes that  $100(1 - \alpha)\%$  confidence bounds for the mean can be derived using the expression

$$\bar{X} \pm \frac{z_{\alpha/2} S}{\sqrt{1000}}$$

where  $\mathbb{P}(Z > z_{\alpha/2}) = \alpha/2$  and  $Z$  is a standard Normal random variable.

What mistake has this student made? Would their technique be valid if they were estimating the service time of this queue?

6. Using the inverse transform method show that

$$X = \left\lfloor \frac{\log(U)}{\log(1-p)} \right\rfloor + 1$$

has a geometric distribution with parameter  $p$  when  $U$  has the  $U(0, 1)$  distribution.

## Examples (ii)

1. Describe a procedure for the generation of the first  $T$  time units of a Poisson process of fixed rate  $\lambda$ .
2. Describe the variance reduction technique based on antithetic variables and give an example of how it is used.
3. For the variance reduction technique based on control variates derive the optimal choice of  $c = c^*$  and its associated variance given in the lectures.
4. Suppose that  $X$  is a random variable having the Binomial distribution with parameters  $n$  and  $p$  and that  $\lambda > 0$  is a constant. Write down the expression for  $\mathbb{P}(X = k)$  where  $k \in \{0, 1, 2, \dots, n\}$ . Now suppose that  $n \rightarrow \infty$  and  $p$  is chosen so that  $p = \lambda/n$ . Show that under this limit  $\mathbb{P}(X = k) \rightarrow e^{-\lambda} \lambda^k / k!$ , that is, to a Poisson distribution with parameter  $\lambda$ .
5. Suppose that  $N(t)$  is the random number of events in the time interval  $[0, t]$  of a Poisson process with parameter  $\lambda$ . State the conditions that define the Poisson process  $N(t)$  and show that for all  $t > 0$  the random variable  $N(t)$  has the Poisson distribution with parameter  $\lambda t$ .
6. In what situations does queueing theory provide appropriate techniques for performance evaluation? When does it not?
7. Show for the M/M/1 queue that the probability that there are  $n$  or more customers in the system is given by  $\rho^n$ .

Use this result to find a service rate  $\mu$  such that, for given  $\lambda, n, \alpha$  where  $0 < \alpha < 1$ , the probability of  $n$  or more customers in the system is given by  $\alpha$ .

Find a value of  $\mu$  for an M/M/1 queue for which the arrival rate is 10 customers per second, and subject to the requirement that the probability of 3 or more customers in the system is 0.05

### Examples (iii)

- Using the steady-state distribution of the number of jobs in a  $M/M/1$  queueing system given in lectures derive the first and second moments of this distribution and hence the variance of the number of jobs present. Describe what happens to the queueing system as the load  $\rho$  increases.
- Given an  $M/M/1/K$  queue with  $\lambda = 10$ ,  $\mu = 12$  and  $K = 15$  over what proportion of time are customers rejected from the queue? What is the effective arrival rate? What is the utilization of the server?
- For the  $M/M/m/m$  loss system show that the loss probability,  $E(\rho, m)$ , that all servers are occupied is given by

$$E(\rho, m) = \frac{\rho^m / m!}{\sum_{i=0}^m \rho^i / i!}$$

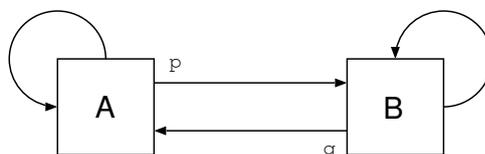
where  $\rho = \lambda/\mu$  is the ratio of the arrival rate to the service rate of a single server. Show also that  $E(\rho, m)$  satisfies the recurrence equation

$$E(\rho, m) = \frac{\rho E(\rho, m-1)}{m + \rho E(\rho, m-1)}$$

with boundary condition  $E(\rho, 0) = 1$ . Why is this expression useful for the numerical computation of loss probabilities?

- Repairing a computer takes 4 stages in sequence, namely removing the lid, finding the faulty part, replacing it, and reassembling the machine. Each step is independent and exponentially distributed with mean 3 minutes. What is the coefficient of variation of the repair time?
- A closed queueing network (shown below) comprises two  $M/M/1$  nodes, **A** and **B** between which  $n$  identical jobs circulate. The nodes have service rates  $\mu_A$  and  $\mu_B$  respectively. Upon completion at **A**, a job moves to **B** with probability  $p$  and otherwise it remains at **A**. Similarly, upon completion at **B**, a job moves to **A** with probability  $q$ .

Derive a Markov-chain model of this system, explaining what each state in the model signifies and what transition rates exist between states.



### **Revision questions for simulation**

1. 2008 Paper 9 Question 12
2. 2007 Paper 7 Question 10
3. 2006 Paper 7 Question 5
4. 2005 Paper 8 Question 16
5. 2003 Paper 8 Question 15

### **Revision questions for queueing theory**

1. 2009 Paper 8 Question 3
2. 2009 Paper 9 Question 4
3. 2008 Paper 8 Question 12
4. 2007 Paper 9 Question 11
5. 2006 Paper 8 Question 11
6. 2005 Paper 7 Question 5
7. 2004 Paper 8 Question 16
8. 2003 Paper 7 Question 5
9. 2002 Paper 8 Question 14
10. 2002 Paper 7 Question 8