Adequacy

$$[\![M]\!] = [\![V]\!] \Longrightarrow M \Downarrow V$$

Induction on M : ?

$$M \equiv M_1(M_2) : \gamma$$

$$M_1 : \tau \to \gamma$$
$$M_2 : \tau$$

$$[\![M_1(M_2)]\!] = [\![V]\!] \overset{?}{\Longrightarrow} M_1(M_2) \Downarrow V$$

$$\cancel{[\![M_1]\!] = \ldots \Rightarrow M_1 \Downarrow \ldots}$$

$M = \underline{fix}(M')$

$[\![fix(M')]\!] = [\![V]\!] \overset{?}{\Longrightarrow} \underline{fix}(M) \Downarrow V$

$\quad \parallel$

$\underline{fix}\,[\![M]\!]$

$[\![M]\!] : [\![\tau]\!] \to [\![\tau]\!]$

$\times$

Logical relations

$[\![M]\!] \lhd_\gamma M \Longrightarrow$ adequacy.

desiderata.

$$\lhd_{nat} \subseteq \mathbb{N}_\perp \times \underline{PCF}_{nat}$$

$$[\![M_1(M_2)]\!] \lhd_\tau M_1(M_2)$$

$$\|$$

$$[\![M_1]\!]([\![M_2]\!])$$

by induction

$$[\![M_1]\!] \lhd_{\tau' \to \tau} M_1$$

$$[\![M_2]\!] \lhd_{\tau'} M_2$$

and would like to conclude

$$[\![M_1]\!]([\![M_2]\!]) \lhd_\tau M_1(M_2)$$

$\boxed{?}$ How should we define

$$\triangleleft_{\tau'\to\tau} \subseteq (\llbracket\tau'\rrbracket \to \llbracket\tau\rrbracket) \times \underline{PCF}_{\tau'\to\tau}$$

?

$$f \triangleleft_{\tau'\to\tau} M$$

iff$_{\text{def}}$ $\forall$ $d \triangleleft_{\tau'} N$.

$$f(d) \triangleleft_{\tau} M(N)$$

$$[\![\underline{fix}(M')]\!] \overset{?}{\lhd}_\tau \underline{fix}(M')$$

$$\|$$

$$fix([\![M']\!]) \qquad\qquad [\![M']\!] : [\![\tau]\!] \to [\![\tau]\!]$$

$$\|$$

$$\bigsqcup_n [\![M']\!]^n(\bot)$$

$(-) \lhd_\tau N$
are admissible.

Show

(1) $- \lhd_{\underline{nat}} \underline{\underline{succ}}^n(\underline{\underline{0}}) \quad \forall n \in \mathbb{N}$

$- \lhd_{\underline{bool}} \underline{\underline{true}}$

$- \lhd_{\underline{bool}} \underline{\underline{false}}$

admissible.

(2) $- \lhd_{\tau' \to \tau} N$ is admissible.

$$[\![\underline{\underline{\mathrm{fix}}}(M')]\!] \lhd_{\tau} \underline{\underline{\mathrm{fix}}}(M')$$

$$[\![M']\!] \lhd_{\tau \to \tau} M'$$

$$\frac{M'(\underline{\underline{\mathrm{fix}}}\, M') \Downarrow V}{\underline{\underline{\mathrm{fix}}}(M') \Downarrow V}$$

$$[\![\underline{\text{fn}}\, x . M]\!] \lhd_{\tau' \to \tau} \underline{\text{fn}}\, x . M$$

$$\|$$

$$[\![x \vdash M]\!]$$

$$[\![M]\!] \lhd_\tau M$$

$$M \in \underline{PCF}_\tau$$

Generalise to a statement involving open terms.

$$\Gamma \equiv x_1 : \tau_1 , \ldots , x_n : \tau_n$$

$$[\![\Gamma \vdash M]\!]$$

$$: [\![\tau_1]\!] \times \cdots \times [\![\tau_n]\!] \longrightarrow [\![\tau]\!]$$

If $d_1 \lhd_{\tau_1} M_1 , \ldots , d_n \lhd_{\tau_n} M_n$

then $\underbrace{[\![\Gamma \vdash M]\!] (d_1 \ldots d_n)}_{\in [\![\tau]\!]} \lhd_\tau \underbrace{M [{}^{M_1}/_{x_1} , \ldots {}^{M_n}/_{x_n}]}_{\in \underline{PCF}_\tau}$

Thm:

$$[\![M_1]\!] \lhd M_2$$

iff $\quad M_1 \leq_{ctx} M_2$

$$M_1 \leq_{ctx} M_2 : \tau_1 \to \tau_2 \to \cdots \to \tau_n \to \gamma$$

iff

$\forall N_1 \cdots N_n$

$$M_1 N_1 \cdots N_n \leq_{ctx} M_2 N_1 \cdots N_n$$

iff $\forall N_1 \cdots N_n$

$$M_1 N_1 \cdots N_n \Downarrow V \Rightarrow M_2 N_1 \cdots N_n \Downarrow V$$