Pre fixed points

EXAMPLES.

$(\mathbb{N}, \leq)$ $(\mathbb{Z}, \leq)$

usual orders.

(1) $\underline{succ} : \mathbb{N} \to \mathbb{N}$

$\underline{succ} : \mathbb{Z} \to \mathbb{Z}$ $\left(: x \mapsto x+1 \right)$

MONOTONICITY

$x \leq y \Rightarrow \underline{succ}(x) = x+1 \leq y+1 = \underline{succ}(y)$

PREFIXED POINTS

x is a pfp iff $\underline{succ}(x) \leq x$

$\therefore$ there are no pfp's for $\underline{succ}$.

2) $\underline{pred} : \mathbb{Z} \to \mathbb{Z}$ $: x \mapsto x-1$

MONOTONICITY

$$x \leq y \Rightarrow \underset{\underset{x-1}{\parallel}}{\underline{pred}(x)} \leq \underset{\underset{y-1}{\parallel}}{\underline{pred}(y)} \quad \checkmark$$

PRE FIXED POINTS

x s.t. $\underline{pred}(x) \leq x$

So every $x \in \mathbb{Z}$ is a pre fixed point.

LEAST PFP

There is none.

(3) $\underline{\text{pred}} : \mathbb{N} \longrightarrow \mathbb{N}$

we want it monotone.

It should be the case that

$$x \le y \Rightarrow \underset{x-1}{\underset{\parallel}{\text{pred}(x)}} \le \underset{y-1}{\underset{\parallel}{\underline{\text{pred}}(y)}}$$

So $\quad 0 \le x \Rightarrow \underline{\text{pred}}(0) \le x-1$

$(\forall x \geq 1)$

forces the definition

$$\underline{\text{pred}}(0) = 0$$

PREFIX POINTS

every $x \in \mathbb{N}$

LEAST PFP

is 0 and in fact it is

a $\underline{\text{fixed point}}$ $\quad \underline{\text{pred}}(0) = 0$

(4) $(\mathcal{P}(\mathbb{N}), \subseteq)$

set of subsets of $\mathbb{N}$

$f : \mathcal{P}(\mathbb{N}) \longrightarrow \mathcal{P}(\mathbb{N})$

$S \subseteq \mathbb{N} \mapsto \{0\} \cup \{x+2 \mid x \in S\}$

$\subseteq \mathbb{N}$

MONOTONICITY

$X \subseteq Y \Rightarrow f(X) \subseteq f(Y)$

Assume $X \subseteq Y$

$f(X) = \{0\} \cup \{x+2 \mid x \in X\}$

$\therefore \cap|$

$f(Y) = \{0\} \cup \{y+2 \mid y \in Y\}$

PREFIXED POINTS

$X \subseteq \mathbb{N}$ s.t. $f(X) \subseteq X$

s.t. $\{0\} \cup \{x+2 \mid x \in X\} \subseteq X$

e.g. $\mathbb{N}$ is a pfp.

$\{0\}$ is not a pfp.

Even is a pfp.

$$\frac{}{0 \in Even} \qquad \frac{x \in Even}{x+2 \in Even}$$

$X \subseteq \mathbb{N}$ is a pfp

iff $0 \in X$

& $\forall x \in X .\ x+2 \in X$

eg. Even $\cup \{k+2n \mid n \in \mathbb{N}\}$

where $k \in$ Odd

THE LEAST PREFIXED POINT EXISTS

and is Even

$f : D \to D$ $\quad$ D poset

L monotone

Assume $\underline{\mathrm{fix}}(f)$, the least prefixed point, exists.

Want to show $f(\underline{\mathrm{fix}}(f)) = \underline{\mathrm{fix}}(f)$.

$$\frac{x \sqsubseteq y}{f(x) \sqsubseteq f(y)} \qquad\qquad \frac{f(d) \sqsubseteq d}{\underline{\mathrm{fix}}(f) \sqsubseteq d}$$

$$\dfrac{\dfrac{}{f(\underline{\mathrm{fix}}(f)) \sqsubseteq \underline{\mathrm{fix}}(f)}\,\mathrm{lpf}1 \qquad \dfrac{\dfrac{\dfrac{}{f(\underline{\mathrm{fix}}\,f) \sqsubseteq \underline{\mathrm{fix}}\,f}\,\mathrm{lpf}1}{f(f(\underline{\mathrm{fix}}\,f)) \sqsubseteq f(\underline{\mathrm{fix}}\,f)}}{\underline{\mathrm{fix}}(f) \sqsubseteq f(\underline{\mathrm{fix}}\,f)}}{f(\underline{\mathrm{fix}}(f)) = \underline{\mathrm{fix}}(f)}$$

• Finite posets

$\boxed{?}$ Yes ~~Is~~ every monotone function on a finite poset with least element has a least prefix point?

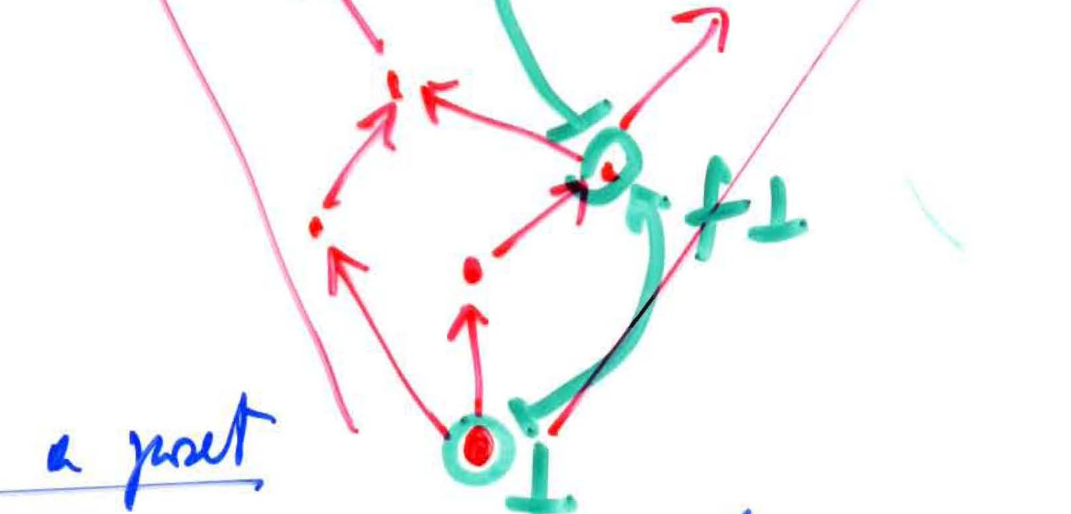
$\mathbb{B} = \{\text{true}, \text{false}\}$

$id : \mathbb{B} \to \mathbb{B}$ $(\mathbb{B}, \sqsubseteq)$

not ~~Swap~~ $: \mathbb{B} \to \mathbb{B}$ $x \sqsubseteq y \text{ iff}_{df} \; x = y$

N.B.

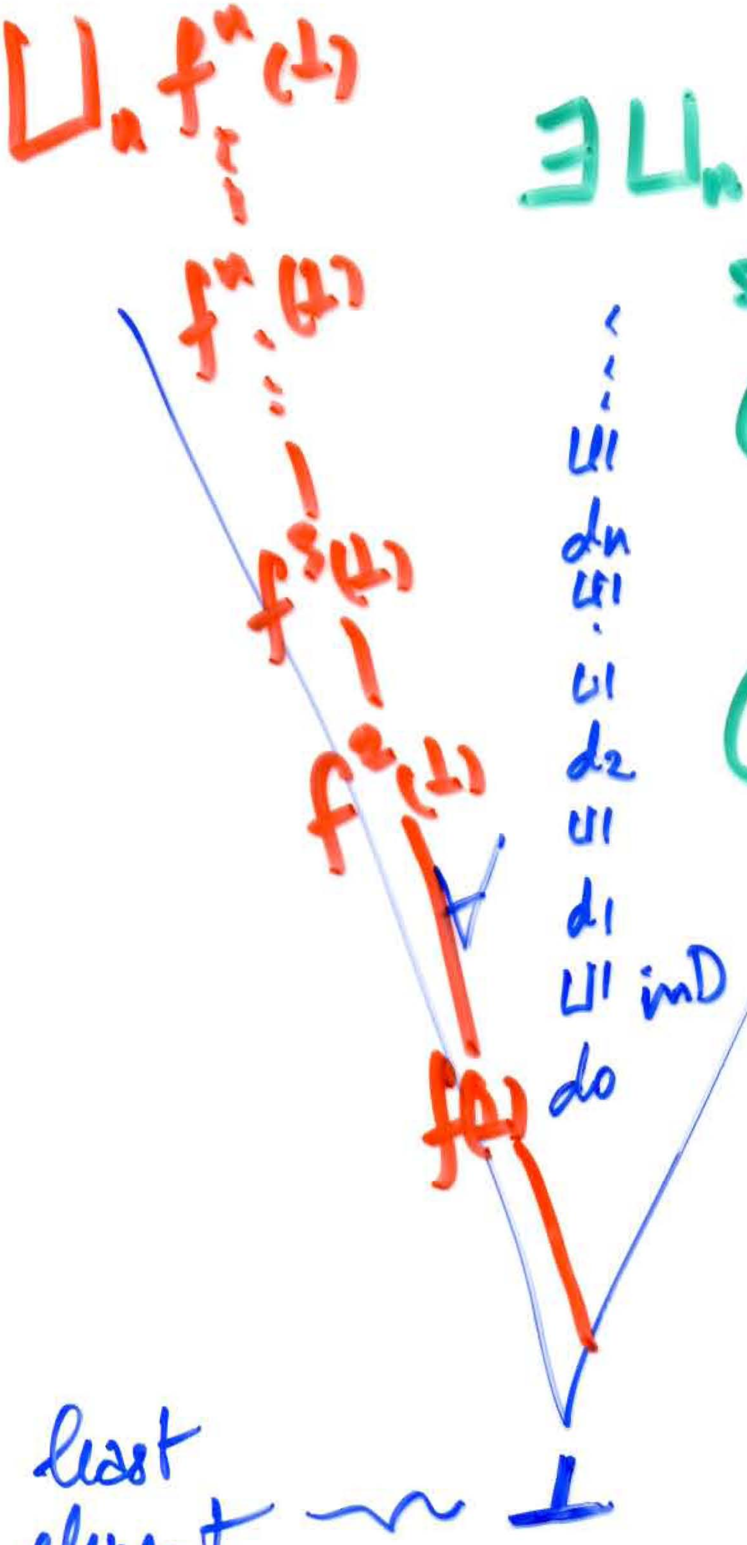
$\perp \sqsubseteq f(\perp) \sqsubseteq \cdots \sqsubseteq f^n(\perp) \sqsubseteq \cdots$



chain in a poset

$x_0 \sqsubseteq x_1 \sqsubseteq \cdots \sqsubseteq x_n \sqsubseteq \cdots \quad \forall n \in \mathbb{N}.$

CPOs



$\exists \bigsqcup_n d_n \in D$ s.t.

(1) $\forall i \in \mathbb{N}.\ d_i \sqsubseteq \bigsqcup_n d_n$

(2) If $d \in D$ s.t. $\forall i \in \mathbb{N}.\ d_i \sqsubseteq d$

Then $\bigsqcup_n d_n \sqsubseteq d$.

D

$f : D \to D$ monotone

IMPORTANT

To define a cpo

(1) give a set. D

(2) give a relation $\sqsubseteq \subseteq D \times D$

(3) show $\sqsubseteq$ is a partial order.

(4) show that there are least upper bounds for all chains.

To define a domain

as above

\+ (5) give me a least element.