$f : D \to D$ cont.

$\underline{fix}(f) = \bigsqcup_n f^n(\bot)$

- $\underline{fix}(f)$ is a prefixed point.
- It is least amongst all prefixed points.

$f(x) \sqsubseteq x \Rightarrow \underline{fix}(f) \sqsubseteq x$

Assume $f(x) \sqsubseteq x$

Show $\underline{fix}(f) \sqsubseteq x$

That is

$$\bigsqcup_n f^n(\bot) \sqsubseteq x$$

$$\dfrac{\dfrac{\dfrac{\checkmark}{\bot \sqsubseteq x} \quad \dfrac{\dfrac{\dfrac{\dfrac{}{\bot \sqsubseteq x}}{f\bot \sqsubseteq f x} \quad f x \sqsubseteq x}{}}{f(\bot) \sqsubseteq x} \quad \ldots f^n\bot \sqsubseteq x \ \ldots}{\forall n. \quad f^n(\bot) \sqsubseteq x} \quad \rightsquigarrow \text{ to be shown by induction on } n}{\bigsqcup_n f^n(\bot) \sqsubseteq x}$$

$$\frac{\forall n \quad x_n \sqsubseteq x}{\bigsqcup_n x_n \sqsubseteq x}$$

$$d_{k,k} \sqsubseteq \bigsqcup_k d_{kk}$$

$$\forall l \quad \overline{x_l \sqsubseteq \bigsqcup_n x_n}$$

$d_{m,n} \sqsubseteq d_{\max(m,n),\max(m,n)} \rightsquigarrow$ by (†) assumption

where $l = \max(m,n)$

$$\frac{\forall m \forall n \quad \overline{d_{m,n} \sqsubseteq d_{l,l}} \quad \overline{d_{ll} \sqsubseteq \bigsqcup_k d_{kk}}}{\dfrac{\forall m \ \forall n \quad d_{m,n} \sqsubseteq \bigsqcup_k d_{kk}}{\dfrac{\forall m \quad \bigsqcup_n d_{m,n} \sqsubseteq \bigsqcup_k d_{kk}}{\bigsqcup_m \left(\bigsqcup_n d_{m,n}\right) \sqsubseteq \bigsqcup_k d_{kk}}}}$$

# ML

int , bool , ...

$$\frac{\tau, \sigma \text{ types}}{\tau * \sigma \text{ type}}$$

$$\frac{\tau, \sigma \text{ types}}{\tau \to \sigma \text{ type}}$$

$\alpha\beta F = \alpha\beta F \to \alpha \to \beta$

recursively defined datatype.

$X$ a set

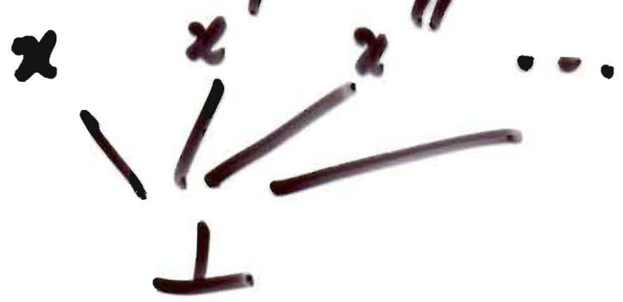
$(X, \sqsubseteq)$ $\quad x \sqsubseteq x'$ iff $x = x'$

$X_\perp = (X \cup \{\perp\}, \sqsubseteq)$

$x \sqsubseteq x'$ iff $x = \perp$ or

$(x \neq \perp \ \& \ x = x')$

$X_\perp =$



$D_1, D_2$ domains.

$D_1 \times D_2$

- has underlying set

  $\{(d_1, d_2) \mid d_1 \in D_1, d_2 \in D_2\}$

- $\sqsubseteq \subseteq (D_1 \times D_2)^2$

  $(d_1, d_2) \sqsubseteq (e_1, e_2)$

  iff $\stackrel{\text{def}}{=}$ $d_1 \sqsubseteq e_1$, $d_2 \sqsubseteq e_2$

- bottom element:

  $(\bot_1, \bot_2)$

- **lubs:**

$$(x_0, y_0) \sqsubseteq (x_1, y_1) \sqsubseteq \cdots \sqsubseteq (x_n, y_n) \sqsubseteq \cdots \quad (*)$$

$\Downarrow$

$x_0 \sqsubseteq x_1 \sqsubseteq \cdots \sqsubseteq x_n \sqsubseteq \cdots$ in $D_1$

$y_0 \sqsubseteq y_1 \sqsubseteq \cdots \sqsubseteq y_n \sqsubseteq \cdots$ in $D_2$

so I can consider

$$\bigsqcup_n x_n \quad \text{and} \quad \bigsqcup_n y_n$$

and show that

$$\left(\bigsqcup_n x_n, \bigsqcup_n y_n\right)$$

is a least upper bound for $(*)$

- Check that for continuous $f : D \times E \to F$:

$$f(\sqcup_n d_n, e) = \sqcup_n f(d_n, e)$$

NB:

$$\boxed{\sqcup_n e = e}$$

Given a chain $(d_n, e_n)$:

$$f(\sqcup_n (d_n, e_n)) = \sqcup_n f(d_n, e_n) \quad (*)$$

by continuity of $f$.

$$\| \; f(\sqcup_n d_n, \sqcup_n e_n)$$

If $e_n = e \; \forall n$

then $(*)$ becomes

$$f(\sqcup_n d_n, \sqcup_n e) = \sqcup_n f(d_n, e)$$

$$\| \; f(\sqcup_n d, e)$$

$$f\left(\bigsqcup_m x_m, \bigsqcup_n y_n\right)$$

$$= \bigsqcup_m f\left(x_m, \bigsqcup_n y_n\right)$$

$$= \bigsqcup_m \bigsqcup_n f(x_m, y_n)$$

$$= \bigsqcup_k f(x_k, y_k)$$

# Function Domains

Given $D$ and $E$ domains.

$(D \to E)$ is defined as follows:

- **underlying set:**

  $\{ f : D \to E \mid f \text{ is continuous} \}$

- **partial order:**

  $f \sqsubseteq_{D\to E} g$ iff$_{def}$ $\forall d \in D.$ $f(d) \sqsubseteq_E g(d)$

- **least element:**

  $\perp_{D\to E}$ is the function defined by setting $\perp_{D\to E}(x) = \perp_E$ $\forall x \in D$.

  NB: That $\perp_{D\to E}$ so defined is continuous.

- **lubs:**

  $f_0 \sqsubseteq f_1 \sqsubseteq \ldots \sqsubseteq f_n \sqsubseteq \ldots$

  what is $\bigsqcup_n f_n$ ?

Define

$(\bigsqcup_n f_n)(x)$ for $x \in D$

$=$

$\bigsqcup_n (f_n(x)) \in E$

$f_0(x) \sqsubseteq f_1(x) \sqsubseteq \cdots \sqsubseteq f_n(x) \sqsubseteq \cdots$

check

(1) it is continuous

(2) it is an upper bound of $\langle f_n \rangle$

(3) it is least.

(1) $\bigsqcup_n f_n$ is continuous

iff monotone &

$(\bigsqcup_n f_n)(\bigsqcup_m x_m)$

$= \bigsqcup_m (\bigsqcup_n f_n)\, x_m$

$x \sqsubseteq x'$

$\Rightarrow (\bigsqcup_n f_n)(x) \sqsubseteq (\bigsqcup_n f_n)(x')$

Assume

$x \sqsubseteq x'$

Show $(\bigsqcup_n f_n)(x) \overset{?}{\sqsubseteq} (\bigsqcup_n f_n)(x')$

$$\begin{array}{ccc} (\bigsqcup_n f_n)(x) & \overset{?}{\sqsubseteq} & (\bigsqcup_n f_n)(x') \\ \| & & \| \\ \bigsqcup_n f_n(x) & & \bigsqcup_n f_n(x') \end{array}$$

$$\begin{array}{ccccccccc} f_0(x') & \sqsubseteq & f_1(x') & \sqsubseteq & \cdots & \sqsubseteq & f_n(x') & \sqsubseteq & \cdots \\ \sqcup\!\!\!\mid & & \sqcup\!\!\!\mid & & & & \sqcup\!\!\!\mid & & \\ f_0(x) & \sqsubseteq & f_1(x) & \sqsubseteq & \cdots & \sqsubseteq & f_n(x) & \sqsubseteq & \cdots \end{array}$$