L108: Category theory and logic
Exercise sheet 2

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1. List all possible functors of types $\mathbf{1} \to \mathbf{Span}$ and $\mathbf{2} \to \mathbf{Span}$ where $\mathbf{1}$ is the category with one object and only the identity morphism, and $\mathbf{2}$ is the category with two objects and one non-identity morphism between them ($\mathbf{Span}$ is defined on the first exercise sheet).

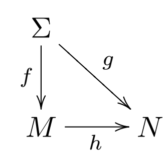
   What are functors of type $\mathbf{1} \to \mathbb{C}$ and $\mathbf{2} \to \mathbb{C}$ for an arbitrary category $\mathbb{C}$?

   Solution. Functors $F : \mathbf{1} \to \mathbb{C}$ are completely determined by $F(*)$, thus they are in bijection with $\mathrm{obj}(\mathbb{C})$. Similarly, functors of type $\mathbf{2} \to \mathbb{C}$ are in bijection with morphisms of $\mathbb{C}$.

   Thus, there are three functors of type $\mathbf{1} \to \mathbf{Span}$ and five functors of type $\mathbf{2} \to \mathbf{Span}$.

2. Let Σ be a set, and let $\Sigma/\mathbf{Mon}$ be the category where

   - *objects* are pairs $((M, \cdot, e), f)$ where $(M, \cdot, e)$ is a monoid and $f : \Sigma \to M$ is a function.
   - *morphisms* from $((M, \cdot, e), f)$ to $((N, \cdot, e), g)$ are monoid homomorphisms $h : (M, \cdot, e) \to (N, \cdot, e)$ such that the triangle

     commutes, i.e. $h \circ f = g$.

   Show that the pair $((\Sigma^*, \cdot, \varepsilon), i)$ is an initial object in $\Sigma/\mathbf{Mon}$, where $(\Sigma^*, \cdot, \varepsilon)$ is the monoid of lists over Σ with concatenation as multiplication and the empty list ε as unit, and $i : \Sigma \to \Sigma^*$ is the function that sends each element $s \in \Sigma$ to the corresponding list $[s]$ of length one.

   Solution. Let $((M, \cdot, e), f)$ be an arbitrary object of $\Sigma/\mathbf{Mon}$. A morphism

   $$g : ((\Sigma^*, \cdot, \varepsilon), i) \to ((M, \cdot, e), f)$$

   is defined by $g(\varepsilon) = e$ and $g([a_1, \ldots, a_n]) = f(a_1) \cdot \ldots \cdot f(a_n)$ for $[a_1, \ldots, a_n] \in \Sigma^*$. For uniqueness, assume that $g, h : (\Sigma^*, \cdot, \varepsilon) \to (M, \cdot, e)$ are two monoid homomorphisms such that $g \circ i = f = h \circ i$. We have to show that $g(s) = h(s)$ for all $s \in \Sigma^*$.

Let $s \in \Sigma^*$ be an arbitrary list. If $s = \varepsilon$, then $g(s) = e = h(s)$, and if $s = [a_1, \ldots, a_n]$, then

$$\begin{aligned} g(s) &= g([a_1, \ldots, a_n]) \\ &= g([a_1]) \cdot \ldots \cdot g([a_n]) \\ &= g(i(a_1)) \cdot \ldots \cdot g(i(a_n)) \\ &= f(a_1) \cdot \ldots \cdot f(a_n) \\ &= h(i(a_1)) \cdot \ldots \cdot h(i(a_n)) \\ &= h([a_1]) \cdot \ldots \cdot h([a_n]) \\ &= h([a_1, \ldots, a_n]) = h(s) \end{aligned}$$

as required.

3. Define a functor List : **Set** $\to$ **Mon** whose object part is given by

$$\text{List}(A) = A^* \qquad \text{(the monoid of lists on } A\text{)}.$$

Prove that your definition is well defined (i.e. verify the axioms in the definition of functor).

Solution. We have to define the 'morphism part' of the functor. Given a function $f : A \to B$, we define $Ff : A^* \to B^*$ by $(Ff)(\varepsilon) = \varepsilon$ and $(Ff)([a_1, \ldots, a_n]) = [f(a_1), \ldots, f(a_n)]$.

First, we have to show that Ff is a monoid homomorphism. Ff preserves the unit element by definition, so it remains to verify the condition about multiplication. For $[a_1, \ldots, a_n], [b_1, \ldots, b_m] \in A^*$ we have:

$$\begin{aligned} (Ff)([a_1, \ldots, a_n]) \cdot (Ff)([b_1, \ldots, b_m]) &= [f(a_1), \ldots, f(a_n)] \cdot [f(b_1), \ldots, f(b_m)] \\ &= [f(a_1), \ldots, f(a_n), f(b_1), \ldots, f(b_m)] \\ &= (Ff)([a_1, \ldots, a_n, b_1, \ldots, b_m] \\ &= (Ff)([a_1, \ldots, a_n] \cdot [b_1, \ldots, b_m]). \end{aligned}$$

To show that F is a functor, it remains to show that $F(\text{id}_A) = id_{A^*}$ and $F(gf) = Fg \circ Ff$ for functions $f : A \to B$ and $g : B \to C$.

For the identity, let $s \in A^*$. If $s = \varepsilon$ we have $F(\text{id}_A)(s) = \varepsilon$, and if $s = [a_1, \ldots, a_n]$ then

$$F(\text{id}_A)(s) = F(\text{id}_A)([a_1, \ldots, a_n]) = [\text{id}(a_1), \ldots, \text{id}(a_n)] = [a_1, \ldots, a_n] = s.$$

Similarly, for $f : A \to B$ and $g : B \to C$ we have

$$Fg(Ff(\varepsilon)) = Fg(\varepsilon) = \varepsilon = F(g \circ f)(\varepsilon) \qquad \text{and}$$
$$Fg(Ff([a_1, \ldots, a_n])) = Fg([fa_1, \ldots, fa_n]) = [g(fa_1), \ldots, g(fa_n)] = F(g \circ f)([a_1, \ldots, a_n].$$

4. Given a set A, a *finite multiset* on A is a function $m : A \to \mathbb{N}$ with $m(a) = 0$ everywhere except for a finite number of $a \in A$. Define $F(A)$ to be the set of finite multisets on A.

(a) Define a structure of commutative monoid on $F(A)$, using the additive monoid structure on $\mathbb{N}$ (in this case, it is more suggestive to write the monoid operation as addition, not as multiplication).

Solution. The sum operation is defined by

$$(m+n)(a) = ma + na \quad \text{for } m, n \in FA \text{ and } a \in A$$

This is well defined since $m+n$ differs from zero only at points where either m or n differs from zero, which can only be the case for finitely many a's. The unit element $e \in FA$ is defined by $e(a) = 0$ for all $a \in A$. The axioms of a commutative monoid are straightforward to verify.

(b) Using this monoid structure, define a functor $F : \mathbf{Set} \to \mathbf{Mon}$.

Solution. Given $f : A \to B$ and $m \in FA$, define $F(f)(m)$ by

$$F(f)(m)(b) = \sum_{a \in f^{-1}(b)} m(a).$$

This sum is well defined since m is different from zero only for finitely many a's, hence all but a finite number of summands is zero as well. To show that this defines a functor, we have to check the following conditions:

i. $F(f)$ well defined as function from FA to FB, i.e. $F(f)(m) \in FB$ for $m \in FA$, i.e. $F(f)(m)(b) \neq 0$ only for finitely many b.
ii. $F(f)$ is a monoid homomorphism
iii. $Fg \circ Ff = F(g \circ f)$ and $F\mathrm{id} = \mathrm{id}$

(i) follows from the facts that $F(f)(m)b$ can only be $\neq 0$ if there exists an $a \in f^{-1}(b)$ with $m(a) \neq 0$, the $f^{-1}(b)$ are disjoint for different b, and the finiteness condition on m.

For condition (ii), that $F(f)$ preserves units is shown as follows:

$$F(f)(e)(b) = \sum_{a \in f^{-1}(b)} e(a) = \sum(0) = 0,$$

and thus $F(f)(e) = e$. For the preservation of sums, let $m, n \in FA$. For $b \in B$ we have

$$\begin{aligned} F(f)(m+n)(b) &= \sum_{a \in f^{-1}(b)} (m+n)(a) \\ &= \sum_{a \in f^{-1}(b)} (ma + na) \\ &= \left(\sum_{a \in f^{-1}(b)} ma \right) + \left(\sum_{a \in f^{-1}(b)} na \right) \\ &= F(f)(m)(b) + F(f)(n)(b). \end{aligned}$$

and thus $F(f)(m+n) = F(f)(m) + F(f)(n)$.

For condition (iii), let $f : A \to B$, $g : B \to C$, $m \in FA$ and $c \in C$. We have to show that $F(g \circ f)(m)(c) = F(g)(F(f)(m))(c)$. The left side is defined as

$$F(g \circ f)(m)(c) = \sum_{a \in (g \circ f)^{-1}(c)} m(a),$$

and the right side rewrites as

$$F(g)(F(f)(m))(c) = \sum_{b \in g^{-1}(c)} F(f)(m)(b) = \sum_{b \in g^{-1}(c)} \sum_{a \in f^{-1}(b)} m(a)$$

It thus remains to show that

$$\sum_{a \in (g \circ f)^{-1}(c)} m(a) = \sum_{\substack{b \in g^{-1}(c) \\ a \in f^{-1}(b)}} m(a)$$

and this follows from the fact that $(g \circ f)^{-1}(c)$ can be written as disjoint union

$$(g \circ f)^{-1}(c) = \dot{\bigcup}_{b \in g^{-1}(c)} f^{-1}(b).$$

Finally, the condition $F\mathrm{id} = \mathrm{id}$ is straightforward.

5. Given a set A, we can define a monoid $(PA, \cup, \varnothing)$, where PA is the power set (the set of all subsets) of A, the monoid operation is given by union $\cup$, with the empty set $\varnothing$ as unit element.

   Define a functor $P : \mathbf{Set} \to \mathbf{Mon}$ whose object part is $P(A) = (PA, \cup, \varnothing)$.

   Solution. For $f : A \to B$, the monoid homomorphism $Pf : P(A) \to P(B)$ defined by $(U \subseteq A) \mapsto f(U) \subseteq P(B)$ where $f(U)$ is defined as $f(U) := \{f(a) \mid a \in U\}$ and is called the *direct image of U under f*. It is easy to see that $f(\varnothing) = \varnothing$; to show that $P(f)$ is a monoid homomorphism it remains thus to show that $P(f)$ commutes with unions. This is shown by the following calculation:

   $$f(U \cup V) = \{f(a) \mid a \in U \vee a \in V\} = \{f(a) \mid a \in U\} \cup \{f(a) \mid a \in V\} = f(U) \cup f(V).$$

   Functoriality of P is easily seen as well:

   $$\begin{aligned} g(f(U)) &= g(\{f(a) \mid a \in U\}) = \{g(f(a)) \mid a \in U\} = (g \circ f)(U) \\ \mathrm{id}(U) &= \{a \mid a \in U\} = U \end{aligned}$$

6. Define natural transformations $\eta : \mathrm{List} \to F$ and $\theta : F \to P$.

   Solution. For a set A and $a \in A$, define δ_a by

   $$\delta_a(a') = \begin{cases} 1 & a = a' \\ 0 & \text{else} \end{cases}$$

   The component at A of the natural transformation $\eta : \mathrm{List} \to F$ is defined as follows:

   $$\begin{aligned} \mathrm{List}(A) &\xrightarrow{\eta_A} F(A) \\ [a_1, \ldots, a_n] &\mapsto \delta_{a_1} + \cdots + \delta_{a_n} \end{aligned}$$

It is easy to show that η_A is a monoid homomorphism. To see that η is a natural transformation it remains to show that the square

$$\begin{array}{ccc} \mathrm{List}(A) & \xrightarrow{\mathrm{List}(f)} & \mathrm{List}(B) \\ {\scriptstyle \eta_A}\downarrow & & \downarrow{\scriptstyle \eta_B} \\ F(A) & \xrightarrow[F(f)]{} & F(B) \end{array}$$

commutes for all functions $f : A \to B$. Let $[a_1, \ldots, a_n] \in \mathrm{List}(A)$. For any $a \in A$ we have $F(f)(\delta_a) = \delta_{f(a)}$, and thus:

$$\eta_B(\mathrm{List}(f)([a_1, \ldots, a_n])) = \eta_B([f(a_1), \ldots, f(a_n)]) = \delta_{f(a_1)} + \cdots + \delta_{f(a_n)} \quad \text{and}$$
$$F(f)(\eta_A([a_1, \ldots, a_n])) = F(f)(\delta_{a_1} + \cdots + \delta_{a_n}) = \delta_{f(a_1)} + \cdots + \delta_{f(a_n)}$$

The natural transformation $\theta : F \to P$ is defined by

$$F(A) \xrightarrow{\theta_A} P(A)$$
$$m \mapsto \{a \in A \mid m(a) \neq 0\}.$$

It is straightforward to see that θ_A is a monoid homomorphism, and it remains to verify the commutativity of the naturality square

$$\begin{array}{ccc} F(A) & \xrightarrow{F(f)} & F(B) \\ {\scriptstyle \theta_A}\downarrow & & \downarrow{\scriptstyle \theta_B} \\ P(a) & \xrightarrow[P(f)]{} & P(B) \end{array}.$$

For $m \in F(A)$ we have

$$\begin{aligned} \theta_B(F(f)(m)) &= \{b \mid F(f)(m)(b) \neq 0\} \\ &= \{b \mid \sum_{a \in f^{-1}(b)} m(a) \neq 0\} \\ &= \{b \mid \exists a \in A \,.\, f(a) = b \wedge m(a) \neq 0\} \\ &= \{f(a) \mid m(a) \neq 0\} \\ &= f(\{a \mid m(a) \neq 0\} \\ &= P(f)(\theta_A(m)). \end{aligned}$$

7. The functor $N : \mathbf{Set} \to \mathbf{Mon}$ is defined by

$$\begin{aligned} N : \mathbf{Set} &\to \mathbf{Mon} \\ A &\mapsto (\mathbb{N}, +, 0) \\ f &\mapsto \mathrm{id}_{\mathbb{N}} \end{aligned}$$

(This is the *constant functor with value* $(\mathbb{N}, +, 0)$.)

Define natural transformations of type $\mathrm{List} \to N$ and $F \to N$ using the length of a list and the 'size' of a multiset.

Solution. Define $\lambda : \text{List} \to N$ by

$$\begin{aligned} \text{List}(A) &\xrightarrow{\lambda_A} (\mathbb{N}, +, 0) \\ [a_1, \dots, a_n] &\mapsto n \end{aligned}$$

It is easy to see that λ_A is a monoid homomorphism. The naturality square for $f : A \to B$ is

$$\begin{array}{ccc} \text{List}(A) & \xrightarrow{\text{List}(f)} & \text{List}(B) \\ {\scriptstyle \lambda_A}\downarrow & & \downarrow{\scriptstyle \lambda_B} \\ (\mathbb{N}, +, 0) & \xrightarrow[\text{id}]{} & (\mathbb{N}, +, 0) \end{array}$$

which commutes because $\text{List}(f)$ preserves the length of lists.