L108: Category theory and logic
Exercise sheet 5

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Finite products

Let $\mathbb{C}$ be a category with finite products.

1. For morphisms $f : A \to B$, $h : B \to X$, $k : B \to Y$ in $\mathbb{C}$, show that

$$\langle h, k\rangle \circ f = \langle h \circ f, k \circ f\rangle.$$

   For morphisms $f : A \to B$, $g : A \to C$, $h : B \to X$, $l : C \to Y$, show that

$$(h \times l) \circ \langle f, g\rangle = \langle h \circ f, l \circ g\rangle.$$

2. For $A \in \mathrm{obj}(\mathbb{C})$, we define the *diagonal map* δ_A by $\delta_A = \langle \mathrm{id}_A, \mathrm{id}_A\rangle : A \to A \times A$. Show that $\delta_B \circ f = (f \times f) \circ \delta_A$ for $f : A \to B$.

3. Show that the mappings $(A, B) \mapsto A \times B$ and $(f, g) \mapsto f \times g$ define a functor $P : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$.

4. Composing the functor P with the diagonal functor $\delta_{\mathbb{C}} : \mathbb{C} \to \mathbb{C} \times \mathbb{C}$, we obtain the functor

$$\begin{aligned} D : \mathbb{C} &\to \mathbb{C} \\ A &\mapsto A \times A \\ f &\mapsto f \times f \end{aligned}$$

   Show that the diagonal mappings δ_A for $A \in \mathbb{C}$ are the components of a natural transormation $\delta : \mathrm{id}_{\mathbb{C}} \to D$. (In particular, draw the relevant naturality square)

   Define natural transformations $\lambda, \rho : D \to \mathrm{id}_{\mathbb{C}}$ whose components are the projections π_1 and π_2, respectively. (Again, verify naturality and draw the relevant square)

Slice categories

Let $\mathbb{C}$ be a category, and $I \in \mathrm{obj}(\mathbb{C})$. The *slice category* $\mathbb{C}/I$ is the category whose

- **objects** are pairs $(A \in \mathrm{obj}(\mathbb{C}), F : A \to I)$, and whose
- **morphisms** from (A, f) to (B, g) are morphisms $h : A \to B$ in $\mathbb{C}$ such that $g \circ h = f$.

$$\begin{tikzcd} A \arrow[rr, "h"] \arrow[dr, "f"'] & & B \arrow[dl, "g"] \\ & I & \end{tikzcd}$$

- Composition and identities are given by composition and identities in $\mathbb{C}$.

5. Show that $\mathbb{C}/I$ has a terminal object. Give a sufficient condition for the existence of initial objects in $\mathbb{C}/I$.

6. Let M be a set. Show that $\mathbf{Set}/M$ has binary (and thus finite) products.

7. Give a characterization of products in $\mathbb{C}/I$, using a concept from the previous exercise sheet.

Coproducts

Coproducts are the dual concept to products. In other words, coproducts in $\mathbb{C}$ are products in $\mathbb{C}^{\mathsf{op}}$. Concretely, a coproduct of objects A, B of a category $\mathbb{C}$ is an object $A + B$[1] together with *injection* maps $\sigma_1 : A \to A + B$, $\sigma_2 : B \to A + B$ such that for all objects Y and morphisms $f : A \to Y$, $g : B \to Y$, there exists a unique $[f, g] : A + B \to Y$ with $[f, g] \circ \sigma_1 = f$ and $[f, g] \circ \sigma_2 = g$.

$$
\begin{tikzcd}
A \arrow[r, "\sigma_1"] \arrow[dr, "f"'] & A+B \arrow[d, dashed, "{[f,g]}"] & B \arrow[l, "\sigma_2"'] \arrow[dl, "g"] \\
& Y &
\end{tikzcd}
$$

8. The *disjoint union* of sets A, B is defined by $A \uplus B = A \times \{0\} \cup B \times \{1\}$. Show that the disjoint union of two sets is a coproduct in $\mathbf{Set}$.

9. Show that $\mathbf{Preord}$ has binary coproducts.

Cartesian closed categories

Let $\mathbb{C}$ be a cartesian closed category.

10. For fixed $C \in \mathrm{obj}(\mathbb{C})$, show that the assignment $B \mapsto C^B$ gives rise to a functor of type $\mathbb{C}^{\mathsf{op}} \to \mathbb{C}$. For this you have to construct the *morphism part* of the functor, i.e. define a function of type
$$\mathbb{C}(A, B) \to \mathbb{C}(C^B, C^A)$$
for all $A, B \in \mathrm{obj}(\mathbb{C})$, and then you have to verify the functor axioms.

10'. Optional: Extending the previous exercise, show that the assignment $(B, C) \mapsto C^B$ gives rise to a functor of type $\mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbb{C}$.

[1] In many (older) books, coproducts are also denoted by $A \coprod B$.