# L11: Algebraic Path Problems with applications to Internet Routing

Lecture 09


Timothy G. Griffin

timothy.griffin@cl.cam.ac.uk
Computer Laboratory
University of Cambridge, UK


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# Dijkstra's algorithm

**Input** : adjacency matrix $\mathbf{A}$ and source vertex $i \in V$,
**Output** : the $i$-th row of $\mathbf{R}$, $\mathbf{R}(i, \_)$.

**begin**
  $S \leftarrow \{i\}$
  $\mathbf{R}(i, i) \leftarrow \overline{1}$
  **for each** $q \in V - \{i\} : \mathbf{R}(i, q) \leftarrow \mathbf{A}(i, q)$
  **while** $S \neq V$
    **begin**
      find $q \in V - S$ such that $\mathbf{R}(i, q)$ is $\leq_{\oplus}^{L}$ -minimal
      $S \leftarrow S \cup \{q\}$
      **for each** $j \in V - S$
        $\mathbf{R}(i, j) \leftarrow \mathbf{R}(i, j) \oplus (\mathbf{R}(i, q) \otimes \mathbf{A}(q, j))$
    **end**
**end**

# Classical proofs of Dijkstra's algorithm (for global optimality) assume

## Semiring Axioms

$$
\begin{array}{rcrcl}
\text{ADD.ASSOCIATIVE} & : & a \oplus (b \oplus c) & = & (a \oplus b) \oplus c \\
\text{ADD.COMMUTATIVE} & : & a \oplus b & = & b \oplus a \\
\text{ADD.LEFT.ID} & : & \overline{0} \oplus a & = & a \\
\text{MULT.ASSOCIATIVE} & : & a \otimes (b \otimes c) & = & (a \otimes b) \otimes c \\
\text{MULT.LEFT.ID} & : & \overline{1} \otimes a & = & a \\
\text{MULT.RIGHT.ID} & : & a \otimes \overline{1} & = & a \\
\text{MULT.LEFT.ANN} & : & \overline{0} \otimes a & = & \overline{0} \\
\text{MULT.RIGHT.ANN} & : & a \otimes \overline{0} & = & \overline{0} \\
\text{L.DISTRIBUTIVE} & : & a \otimes (b \oplus c) & = & (a \otimes b) \oplus (a \otimes c) \\
\text{R.DISTRIBUTIVE} & : & (a \oplus b) \otimes c & = & (a \otimes c) \oplus (b \otimes c)
\end{array}
$$

# Classical proofs of Dijkstra's algorithm assume

## Additional axioms

$$\begin{array}{rcrcl} \text{ADD.SELECTIVE} & : & a \oplus b & \in & \{a,\ b\} \\ \text{ADD.ANN} & : & \overline{1} \oplus a & = & \overline{1} \end{array}$$

Note that we can derive

$$\text{RIGHT.ABSORBTION} \quad : \quad a \oplus (a \otimes b) \quad = \quad a$$

and this gives (right) inflationarity, $\forall a, b : a \leq a \otimes b$.

# Our goal will be simpler

## Theorem 9.1

Given adjacency matrix **A** and source vertex $i \in V$, Dijkstra's algorithm will compute $\mathbf{R}(i, \_)$ such that

$$\forall j \in V : \mathbf{R}(i, j) = \mathbf{I}(i,j) \oplus \bigoplus_{q \in V} \mathbf{R}(i, q) \otimes \mathbf{A}(q, j).$$

That is, it computes one row of the solution for the right equation

$$\mathbf{X} = \mathbf{X}\mathbf{A} \oplus \mathbf{I}.$$

# What will we assume?

## ~~Semiring~~ Axioms

| | | | | |
|---|---|---|---|---|
| ADD.ASSOCIATIVE | : | $a \oplus (b \oplus c)$ | $=$ | $(a \oplus b) \oplus c$ |
| ADD.COMMUTATIVE | : | $a \oplus b$ | $=$ | $b \oplus a$ |
| ADD.LEFT.ID | : | $\overline{0} \oplus a$ | $=$ | $a$ |
| ~~MULT.ASSOCIATIVE~~ | : | ~~$a \otimes (b \otimes c)$~~ | ~~$=$~~ | ~~$(a \otimes b) \otimes c$~~ |
| MULT.LEFT.ID | : | $\overline{1} \otimes a$ | $=$ | $a$ |
| ~~MULT.RIGHT.ID~~ | : | ~~$a \otimes \overline{1}$~~ | ~~$=$~~ | ~~$a$~~ |
| ~~MULT.LEFT.ANN~~ | : | ~~$\overline{0} \otimes a$~~ | ~~$=$~~ | ~~$\overline{0}$~~ |
| ~~MULT.RIGHT.ANN~~ | : | ~~$a \otimes \overline{0}$~~ | ~~$=$~~ | ~~$\overline{0}$~~ |
| ~~L.DISTRIBUTIVE~~ | : | ~~$a \otimes (b \oplus c)$~~ | ~~$=$~~ | ~~$(a \otimes b) \oplus (a \otimes c)$~~ |
| ~~R.DISTRIBUTIVE~~ | : | ~~$(a \oplus b) \otimes c$~~ | ~~$=$~~ | ~~$(a \otimes c) \oplus (b \otimes c)$~~ |

# What will we assume?

## Additional axioms

$$
\begin{array}{rcrcl}
\text{ADD.SELECTIVE} & : & a \oplus b & \in & \{a,\ b\} \\
\text{ADD.ANN} & : & \overline{1} \oplus a & = & \overline{1} \\
\text{RIGHT.ABSORBTION} & : & a \oplus (a \otimes b) & = & a
\end{array}
$$

Note that we can no longer derive RIGHT.ABSORBTION, so we must assume it.

# Dijkstra's algorithm, annotated version

Subscripts make proofs by induction easier ....

**begin**
  $S_1 \leftarrow \{i\}$
  $\mathbf{R}_1(i,\ i) \leftarrow \overline{1}$
  **for each** $q \in V - S_1$ : $\mathbf{R}_1(i,\ q) \leftarrow \mathbf{A}(i,\ q)$
  **for each** $k = 2, 3, \ldots, \mid V \mid$
    **begin**
      find $q_k \in V - S_{k-1}$ such that $\mathbf{R}(i,\ q)$ is $\leq^L_\oplus$ -minimal
      $S_k \leftarrow S_{k-1} \cup \{q_k\}$
      **for each** $j \in V - S_k$
        $\mathbf{R}_k(i,\ j) \leftarrow \mathbf{R}_{k-1}(i,\ j) \oplus (\mathbf{R}_{k-1}(i,\ q_k) \otimes \mathbf{A}(q_k,\ j))$
    **end**
**end**

# On to the proof ...

## Main Claim

$$\forall k : 1 \leq k \leq \mid V \mid \implies \forall j \in S_k : \mathbf{R}_k(i,\ j) = \mathbf{I}(i, j) \oplus \bigoplus_{q \in S_k} \mathbf{R}_k(i,\ q) \otimes \mathbf{A}(q,\ j)$$

## Observation 1

$$\forall k : 1 \leq k < \mid V \mid \implies \forall j \in S_{k+1} : \mathbf{R}_k(i,\ j) = \mathbf{R}_{k+1}(i,\ j)$$

This is easy to see — once a node is put into $S$ its weight never changes.

# Observation 2

## Observation 2

$$\forall k : 1 \leq k \leq \mid V \mid \implies \forall q \in S_k : \forall w \in V - S_k : \mathbf{R}_k(i,\ q) \leq \mathbf{R}_k(i,\ w)$$

By induction.
Base : Need $\overline{1} \leq \mathbf{A}(i,\ w)$. OK
Induction. Assume

$$\forall q \in S_k : \forall w \in V - S_k : \mathbf{R}_k(i,\ q) \leq \mathbf{R}_k(i,\ w)$$

and show

$$\forall q \in S_{k+1} : \forall w \in V - S_{k+1} : \mathbf{R}_{k+1}(i,\ q) \leq \mathbf{R}_{k+1}(i,\ w)$$

Since $S_{k+1} = S_k \cup \{q_{k+1}\}$, this is means showing

$$\begin{array}{ll} (1) & \forall q \in S_k : \forall w \in V - S_{k+1} : \mathbf{R}_{k+1}(i,\ q) \leq \mathbf{R}_{k+1}(i,\ w) \\ (2) & \forall w \in V - S_{k+1} : \mathbf{R}_{k+1}(i,\ q_{k+1}) \leq \mathbf{R}_{k+1}(i,\ w) \end{array}$$

By Observation 1, showing (1) is the same as

$$\forall q \in S_k : \forall w \in V - S_{k+1} : \mathbf{R}_k(i,\ q) \leq \mathbf{R}_{k+1}(i,\ w)$$

which expands to (by definition of $\mathbf{R}_{k+1}(i,\ w)$)

$$\forall q \in S_k : \forall w \in V - S_{k+1} : \mathbf{R}_k(i,\ q) \leq \mathbf{R}_k(i,\ w) \oplus (\mathbf{R}_k(i,\ q_{k+1}) \otimes \mathbf{A}(q_{k+1},\ w))$$

But $\mathbf{R}_k(i,\ q) \leq \mathbf{R}_k(i,\ w)$ by the induction hypothesis, and $\mathbf{R}_k(i,\ q) \leq (\mathbf{R}_k(i,\ q_{k+1}) \otimes \mathbf{A}(q_{k+1},\ w))$ by the induction hypothesis and RINF.

Since $a \leq^L_\oplus b \wedge a \leq^L_\oplus c \implies a \leq^L_\oplus (b \oplus c)$, we are done.

By Observation 1, showing (2) is the same as showing

$$\forall w \in V - S_{k+1} : \mathbf{R}_k(i,\ q_{k+1}) \leq \mathbf{R}_{k+1}(i,\ w)$$

which expands to

$$\forall w \in V - S_{k+1} : \mathbf{R}_k(i,\ q_{k+1}) \leq \mathbf{R}_k(i,\ w) \oplus (\mathbf{R}_k(i,\ q_{k+1}) \otimes \mathbf{A}(q_{k+1},\ w))$$

But $\mathbf{R}_k(i,\ q_{k+1}) \leq \mathbf{R}_k(i,\ w)$ since $q_{k+1}$ was chosen to be minimal, and $\mathbf{R}_k(i,\ q_{k+1}) \leq (\mathbf{R}_k(i,\ q_{k+1}) \otimes \mathbf{A}(q_{k+1},\ w))$ by RINF.
Since $a \leq^L_\oplus b \wedge a \leq^L_\oplus c \implies a \leq^L_\oplus (b \oplus c)$, we are done.

# Observation 3

## Observation 3

$$\forall k : 1 \leq k \leq \mid V \mid \Longrightarrow \forall w \in V - S_k : \mathbf{R}_k(i,\ w) = \bigoplus_{q \in S_k} \mathbf{R}_k(i,\ q) \otimes \mathbf{A}(q,\ w)$$

Proof: By induction:

Base : easy, since

$$\bigoplus_{q \in S_1} \mathbf{R}_1(i,\ q) \otimes \mathbf{A}(q,\ w) = \overline{1} \otimes \mathbf{A}(i,\ w) = \mathbf{A}(i,\ w) = \mathbf{R}_1(i,\ w)$$

Induction step. Assume

$$\forall w \in V - S_k : \mathbf{R}_k(i,\ w) = \bigoplus_{q \in S_k} \mathbf{R}_k(i,\ q) \otimes \mathbf{A}(q,\ w)$$

and show

$$\forall w \in V - S_{k+1} : \mathbf{R}_{k+1}(i,\ w) = \bigoplus_{q \in S_{k+1}} \mathbf{R}_{k+1}(i,\ q) \otimes \mathbf{A}(q,\ w)$$

By Observation 1, and a bit of rewriting, this means we must show

$$\forall w \in V - S_{k+1} : \mathbf{R}_{k+1}(i,\ w) = \mathbf{R}_k(i,\ q_{k+1}) \otimes \mathbf{A}(q_{k+1},\ w) \oplus \bigoplus_{q \in S_k} \mathbf{R}_k(i,\ q) \otimes \mathbf{A}($$

Using the induction hypothesis, this becomes

$$\forall w \in V - S_{k+1} : \mathbf{R}_{k+1}(i,\ w) = \mathbf{R}_k(i,\ q_{k+1}) \otimes \mathbf{A}(q_{k+1},\ w) \oplus \mathbf{R}_k(i,\ w)$$

But this is exactly how $\mathbf{R}_{k+1}(i,\ w)$ is computed in the algorithm.

# Proof of Main Claim

## Main Claim

$$\forall k : 1 \leq k \leq | V | \Longrightarrow \forall j \in S_k : \mathbf{R}_k(i,\ j) = \mathbf{I}(i, j) \oplus \bigoplus_{q \in S_k} \mathbf{R}_k(i,\ q) \otimes \mathbf{A}(q,\ j)$$

Proof : By induction on $k$.
Base case: $S_1 = \{i\}$ and the claim is easy.
Induction: Assume that

$$\forall j \in S_k : \mathbf{R}_k(i,\ j) = \mathbf{I}(i, j) \oplus \bigoplus_{q \in S_k} \mathbf{R}_k(i,\ q) \otimes \mathbf{A}(q,\ j)$$

We must show that

$$\forall j \in S_{k+1} : \mathbf{R}_{k+1}(i,\ j) = \mathbf{I}(i, j) \oplus \bigoplus_{q \in S_{k+1}} \mathbf{R}_{k+1}(i,\ q) \otimes \mathbf{A}(q,\ j)$$

Since $S_{k+1} = S_k \cup \{q_{k+1}\}$, this means we must show

$$\begin{array}{ll} (1) & \forall j \in S_k : \mathbf{R}_{k+1}(i,\ j) = \mathbf{I}(i,j) \oplus \bigoplus_{q \in S_{k+1}} \mathbf{R}_{k+1}(i,\ q) \otimes \mathbf{A}(q,\ j) \\ (2) & \mathbf{R}_{k+1}(i,\ q_{k+1}) = \mathbf{I}(i,q_{k+1}) \oplus \bigoplus_{q \in S_{k+1}} \mathbf{R}_{k+1}(i,\ q) \otimes \mathbf{A}(q,\ q_{k+1}) \end{array}$$

By use Observation 1, showing (1) is the same as showing

$$\forall j \in S_k : \mathbf{R}_k(i,\ j) = \mathbf{I}(i,j) \oplus \bigoplus_{q \in S_{k+1}} \mathbf{R}_k(i,\ q) \otimes \mathbf{A}(q,\ j),$$

which is equivalent to

$$\forall j \in S_k : \mathbf{R}_k(i,\ j) = \mathbf{I}(i,j) \oplus (\mathbf{R}_k(i,\ q_{k+1}) \otimes \mathbf{A}(q_{k+1},\ j)), \oplus \bigoplus_{q \in S_k} \mathbf{R}_k(i,\ q) \otimes \mathbf{A}(q,$$

By the induction hypothesis, this is equivalent to

$$\forall j \in S_k : \mathbf{R}_k(i,\ j) = \mathbf{R}_k(i,\ j) \oplus (\mathbf{R}_k(i,\ q_{k+1}) \otimes \mathbf{A}(q_{k+1},\ j)),$$

Put another way,

$$\forall j \in S_k : \mathbf{R}_k(i,\ j) \le \mathbf{R}_k(i,\ q_{k+1}) \otimes \mathbf{A}(q_{k+1},\ j)$$

By observation 2 we know $\mathbf{R}_k(i,\ j) \le \mathbf{R}_k(i,\ q_{k+1})$, and so

$$\mathbf{R}_k(i,\ j) \le \mathbf{R}_k(i,\ q_{k+1}) \le \mathbf{R}_k(i,\ q_{k+1}) \otimes \mathbf{A}(q_{k+1},\ j)$$

by RINF.

To show (2), we use Observation 1 and $\mathbf{I}(i, q_{k+1}) = \overline{0}$ to obtain

$$\mathbf{R}_k(i,\ q_{k+1}) = \bigoplus_{q \in S_{k+1}} \mathbf{R}_k(i,\ q) \otimes \mathbf{A}(q,\ q_{k+1})$$

which, since $\mathbf{A}(q_{k+1},\ q_{k+1}) = \overline{0}$, is the same as

$$\mathbf{R}_k(i,\ q_{k+1}) = \bigoplus_{q \in S_k} \mathbf{R}_k(i,\ q) \otimes \mathbf{A}(q,\ q_{k+1})$$

This then follows directly from Observation 3.

# Finding Left Local Solutions?

$$\mathbf{L} = (\mathbf{A} \otimes \mathbf{L}) \oplus \mathbf{I} \quad \iff \quad \mathbf{L}^T = (\mathbf{L}^T \otimes^T \mathbf{A}^T) \oplus \mathbf{I}$$

$$\mathbf{R}^T = (\mathbf{A}^T \otimes^T \mathbf{R}^T) \oplus \mathbf{I} \quad \iff \quad \mathbf{R} = (\mathbf{R} \otimes \mathbf{A}) \oplus \mathbf{I}$$

where

$$a \otimes^T b = b \otimes a$$

Notice that this exchanges RINF for LINF!

$$\text{LINF} : \forall a, b : a \leq b \otimes a$$

# Notes

- Complexity of solving for left local optima?
  - Previous work has shown that Bellman-Ford will find a solution as long as only simple paths are explored — but no time bounds are known.
  - Dijkstra's algorithm : $O(V^3)$
  - Could do better in sparse graphs using Fibonacci heaps ...

# HW 2 : Recall a few definitions

## Recall definition of a reduction

If $S \equiv (S, \oplus, \otimes)$ is a semiring and $r$ is a function from $S$ to $S$, then $r$ is a reduction for $S$ if for all $a$ and $b$ in $S$

1. $r(a) = r(r(a))$
2. $r(a \oplus b) = r(r(a) \oplus b) = r(a \oplus r(b))$
3. $r(a \otimes b) = r(r(a) \otimes b) = r(a \otimes r(b))$

## Reduce operation

If $(S, \oplus, \otimes)$ is semiring and $r$ is a reduction, then let $\mathrm{red}_r(S) = (S_r, \oplus_r, \otimes_r)$ where

1. $S_r = \{s \in S \mid r(s) = s\}$
2. $x \oplus_r y = r(x \oplus y)$
3. $x \otimes_r y = r(x \otimes y)$

# HW2 : A few more definitions

## Recall: Lifted product semiring

Assume $(S, \otimes, \overline{1})$ is a monoid (a semigroup with identity $\overline{1}$). Define the semiring

$$\text{lift}(S) = (\mathcal{P}_{\text{fin}}(S), \cup, \hat{\otimes}, \{\}, \{\overline{1}\})$$

where

$$X \hat{\otimes} Y = \{x \otimes y \mid x \in X, \ y \in Y\}$$

for $X, Y \in \mathcal{P}_{\text{fin}}(S)$, the set of finite subsets of $S$.

## Definition: min-sets

Suppose that $(S, \leq)$ is a pre-ordered set (reflexive, transitive pre-order). Let $A \subseteq S$ be finite. Define

$$\min_{\leq}(A) \equiv \{a \in A \mid \forall b \in A : \neg(b < a)\}$$

# HW 2 : Questions 1 and 2

## Question 1 (30 points)

Is it always the case that $\mathrm{red}_r(S)$ is a semiring? If so prove this. Otherwise impose some conditions on $r$ that would guarantee that $\mathrm{red}_r(S)$ is a semiring.

## Question 2 (40 points)

Is $\min_{\leq}$ always a reduction for the semiring $\mathrm{lift}(S)$? If not, impose some constraints on $\leq$ and $\otimes$ that will result in $\min_{\leq}$ being a reduction.

# HW2 : Question 3

The lecture notes introduced this "reduction"

$$
\begin{array}{rcl}
r(\infty) & = & \infty \\
r(s, W) & = & \begin{cases} \infty & \text{if } W = \{\} \\ (s, W) & \text{otherwise} \end{cases}
\end{array}
$$

and then gave an example using the algebra

$$\mathrm{s} \equiv \mathrm{red}_r(\mathrm{add\_zero}(\infty,\ \mathrm{min\_plus} \vec{\times} \mathrm{sep}(G)))$$

## Question 3 (30 points)

(a) Show that $r$ is not in fact a reduction.

(b) Suppose that $\mathbf{A}$ is an adjacency matrix over algebra $\mathrm{s}$. Looking only at the first component of the metric, suppose there are no 0-weight cycles in the graph. Argue that starting with any $\mathbf{M}$ and iterating using $\mathbf{A}_{\mathbf{M}}^{\langle k \rangle}$ we will arrive at $\mathbf{A}^*$.