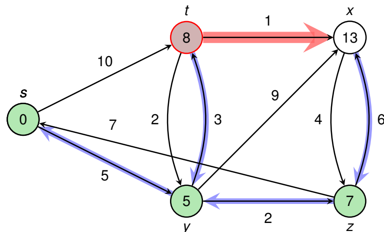

6.4: Single-Source Shortest Paths

Frank Stajano

Thomas Sauerwald

Lent 2015

UNIVERSITY OF CAMBRIDGE

Outline

Bellman-Ford

Dijkstra's Algorithm

All-Pairs Shortest Path

APSP via Matrix Multiplication

Johnson's Algorithm

The Bellman-Ford Algorithm

```
 BELLMAN-FORD(G,w,s)
0: assert(s in G.vertices())
1: for v in G.vertices()
2:    v.predecessor = None
3:    v.d = Infinity
4: s.d = 0
5:
6: repeat |V|-1 times
7:    for e in G.edges()
8:    Relax edge e=(u,v):  Check if u.d + w(u,v) < v.d
9:       if e.start.d + e.weight.d < e.end.d:
10:         e.end.d = e.start.d + e.weight
11:         e.end.predecessor = e.start
12:
13: for e in G.edges()
14:    if e.start.d + e.weight.d < e.end.d:
15:       return FALSE
16: return TRUE
```

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Can we terminate earlier if there is a pass that keeps all *.d* variables?

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Yes, because if pass i keeps all *.d* variables, then so does pass $i + 1$.

The Bellman-Ford Algorithm (modified)

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 BELLMAN-FORD-NEW(G,w,s)
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5:
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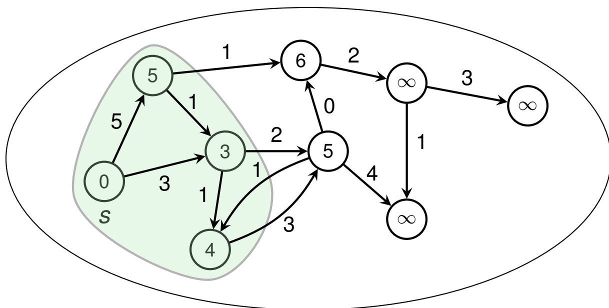

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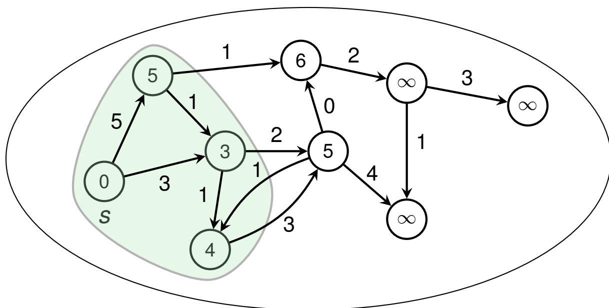

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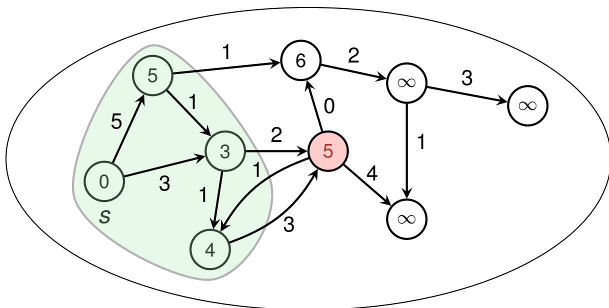

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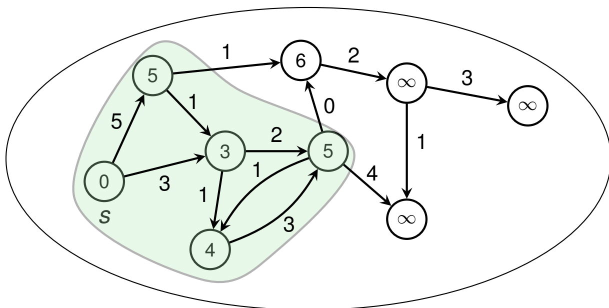

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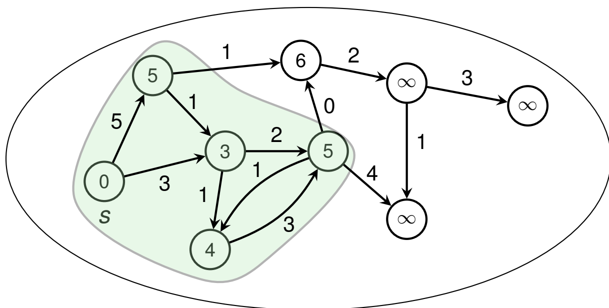

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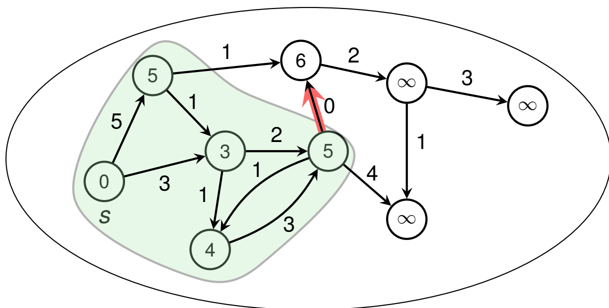

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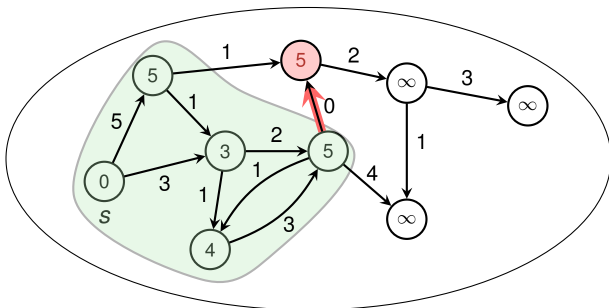

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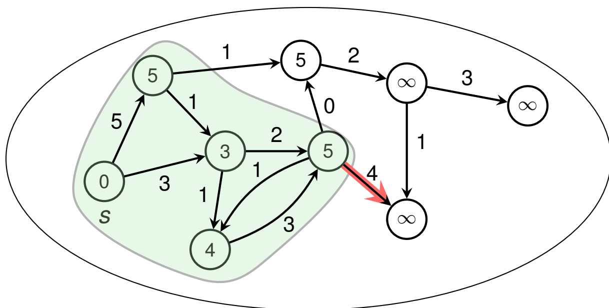

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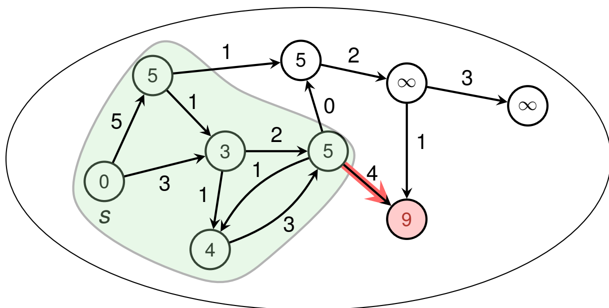

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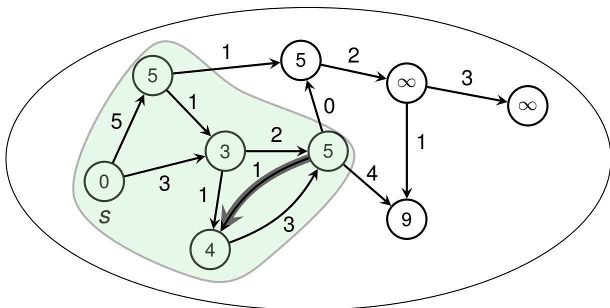

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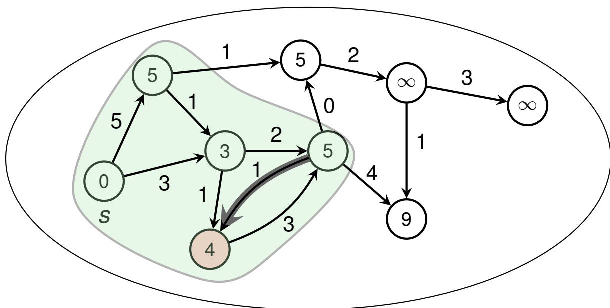

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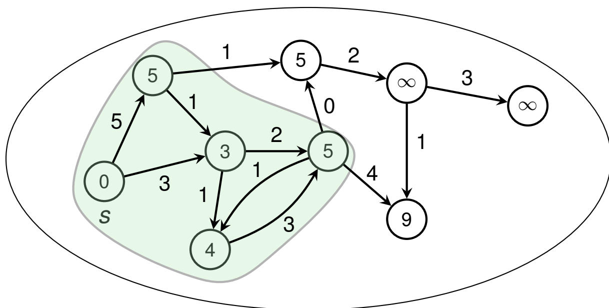

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DIJKSTRA(G,w,s)

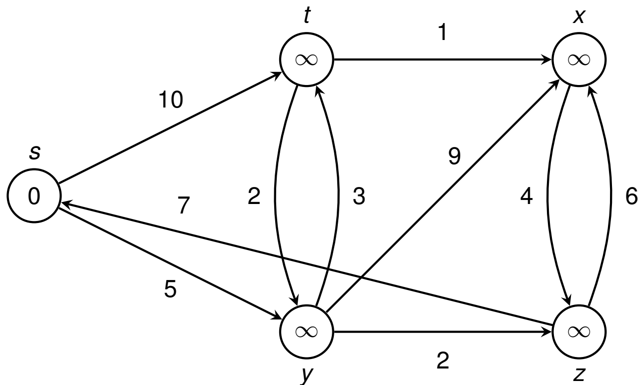
0: INITIALIZE(G,s)
1: $S = \emptyset$
2: $Q = V$
3: **while** $Q \neq \emptyset$ **do**
4:     u = Extract-Min(Q)
5:     $S = S \cup \{u\}$
6:     **for** each $v \in G.Adj[u]$ **do**
7:         RELAX(u, v, w)
8:     **end for**
9: **end while**

can actually skip edges which go to a vertex in S (cf. implementation of Dijksta in the handout)

Complete Run of Dijkstra (Figure 24.6)

Priority Queue Q:

$(s, 0), (t, \infty), (x, \infty), (y, \infty), (z, \infty)$



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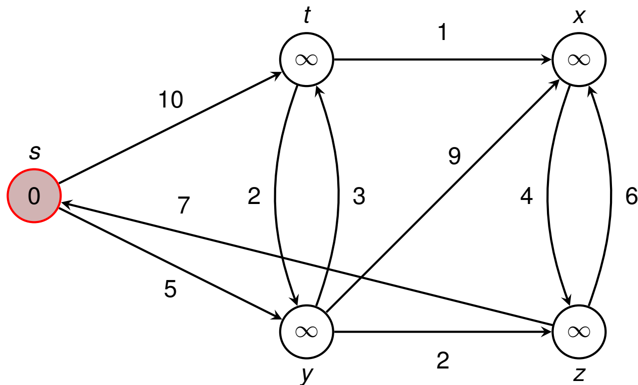

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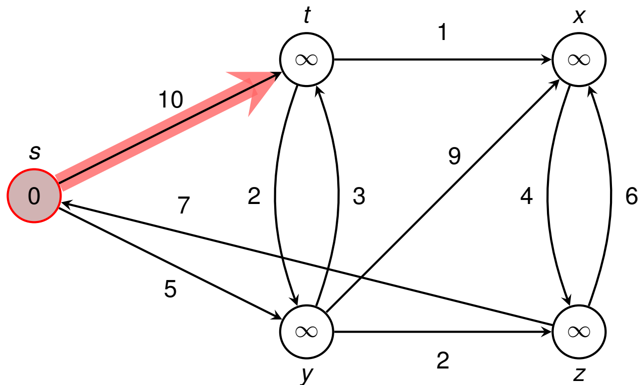

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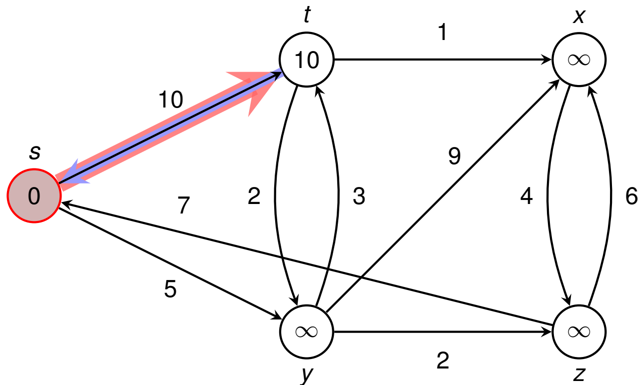

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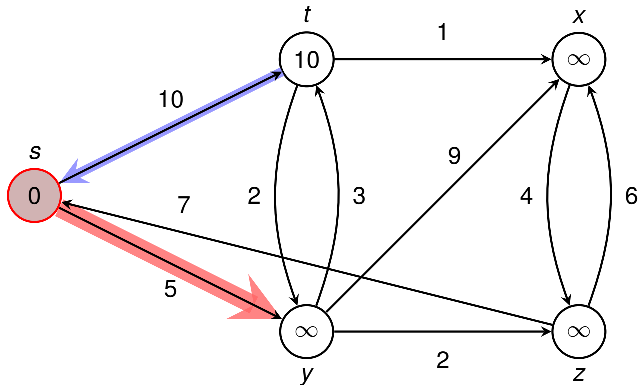

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Priority Queue Q:

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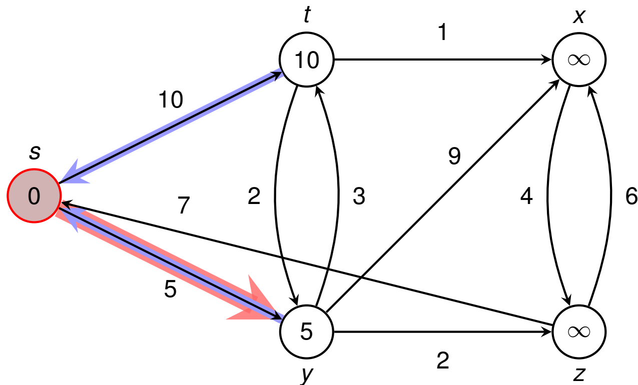

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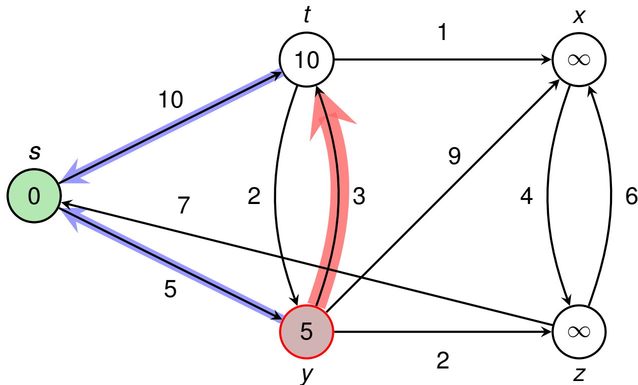

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Priority Queue Q:

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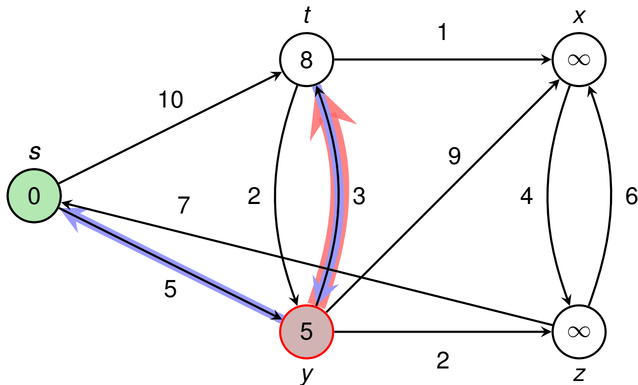

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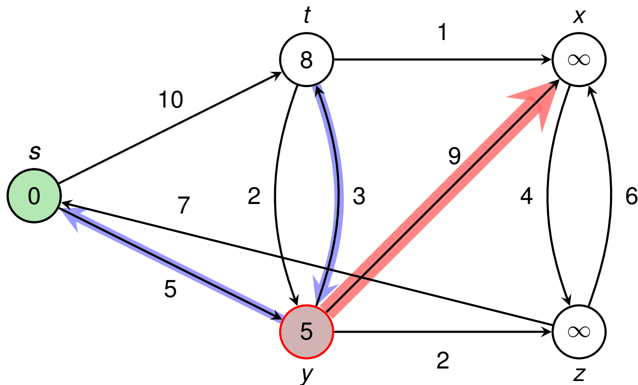

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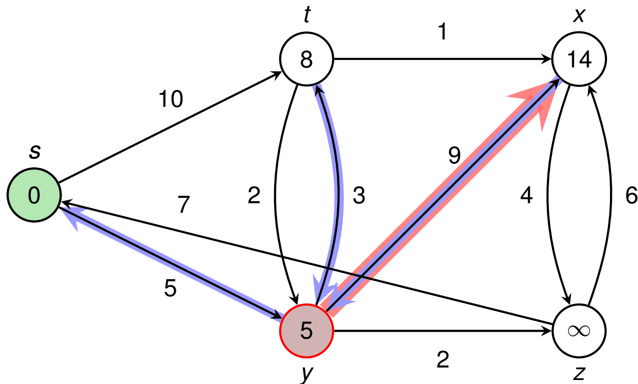

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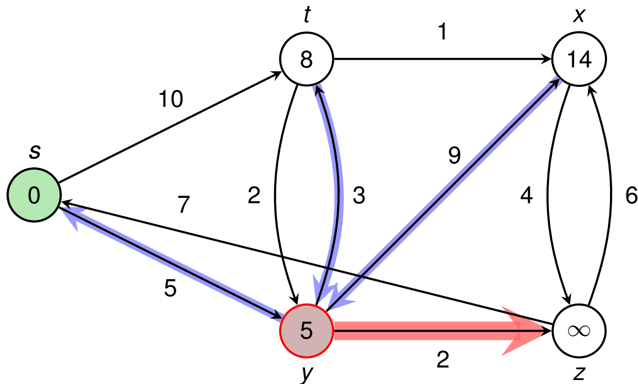

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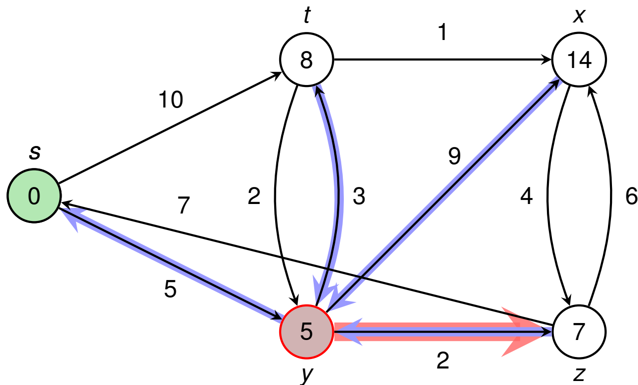

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Priority Queue Q:

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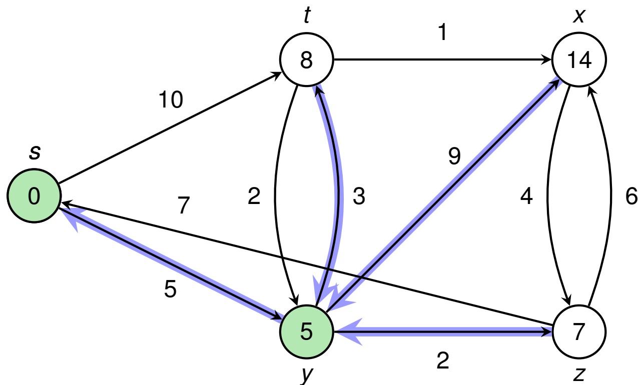

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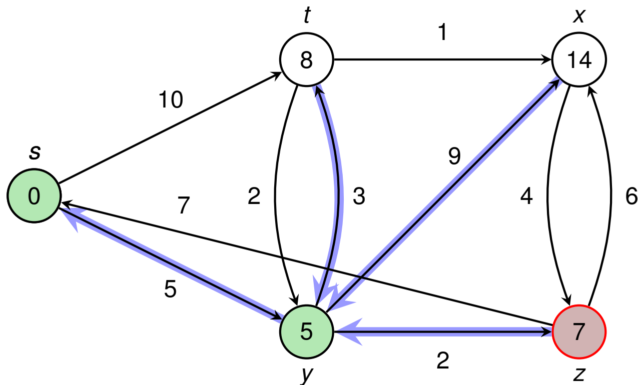

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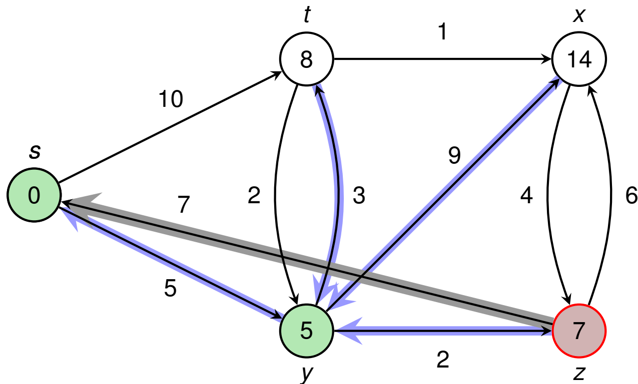

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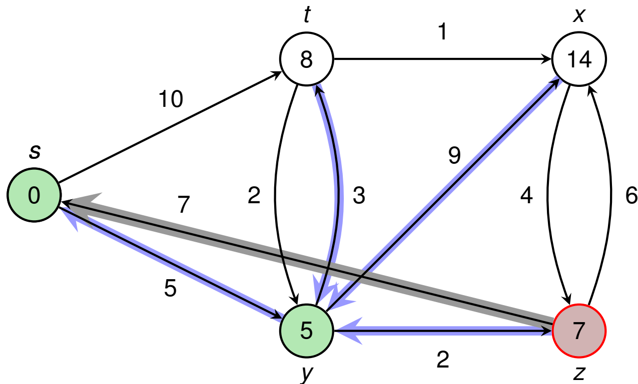

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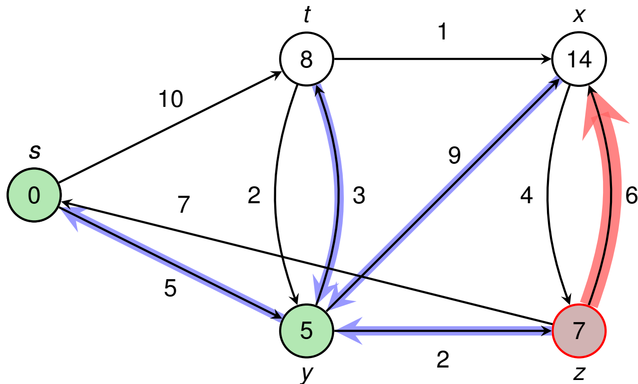

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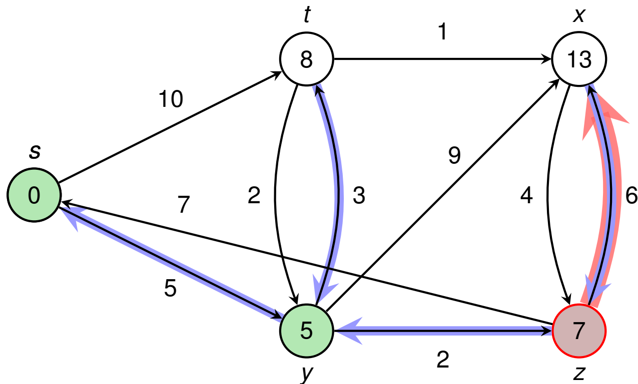

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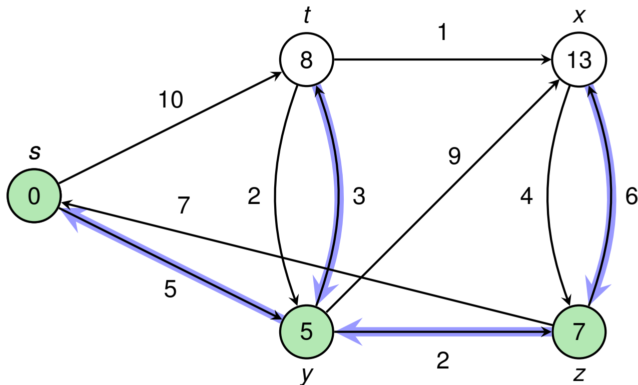

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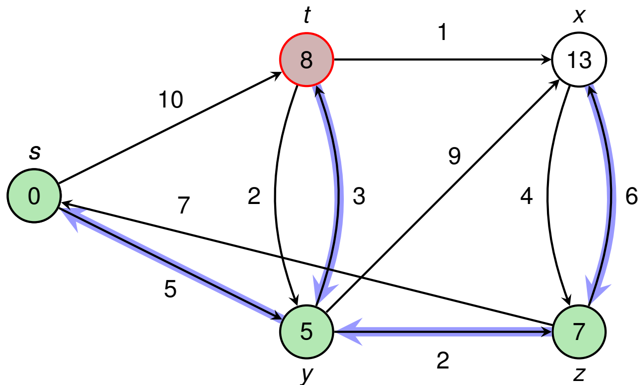

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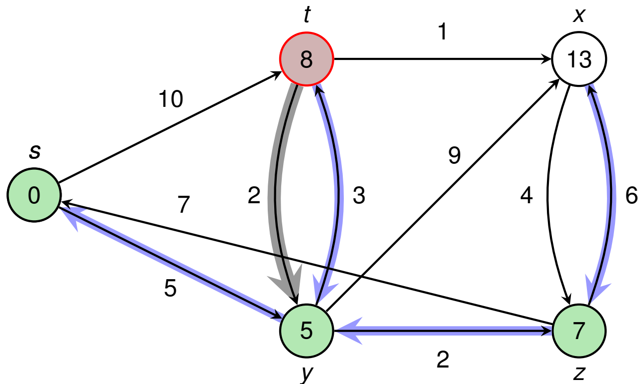

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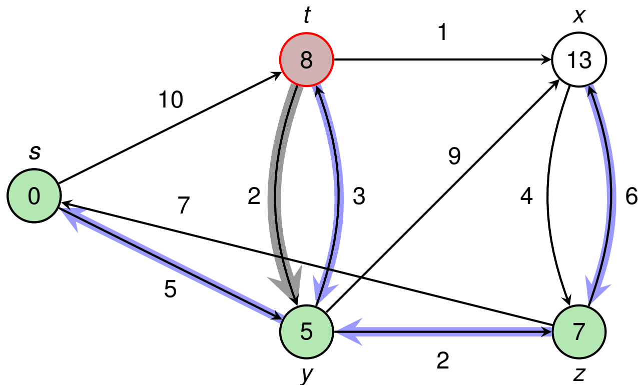

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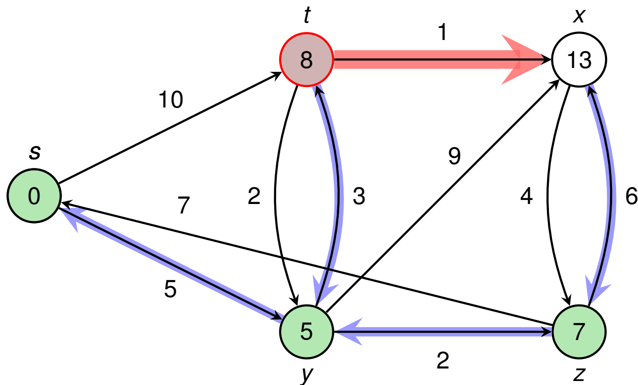

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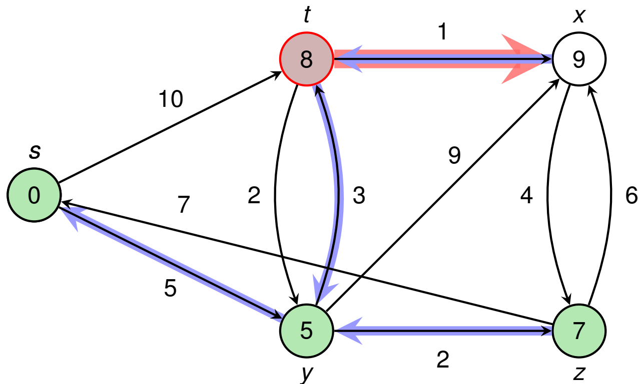

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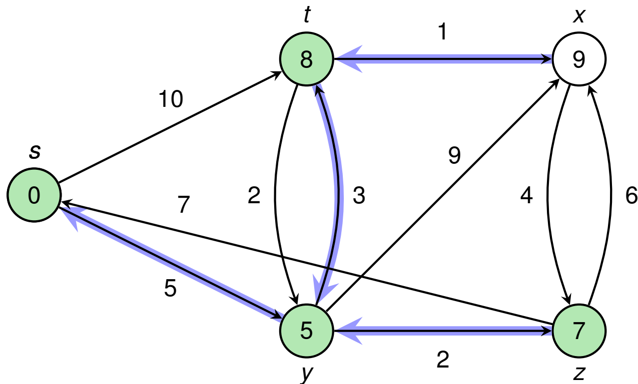

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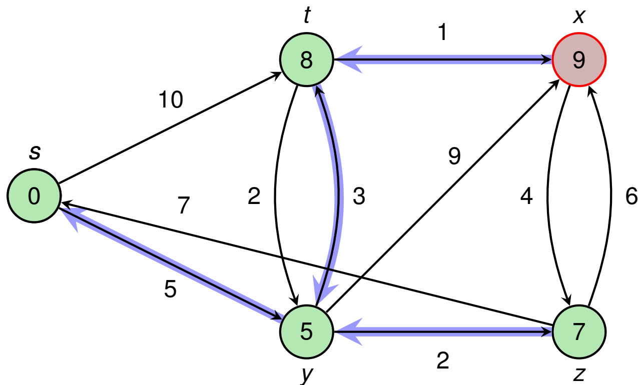

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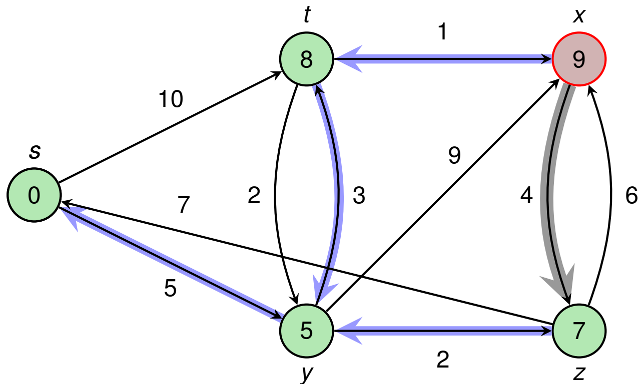

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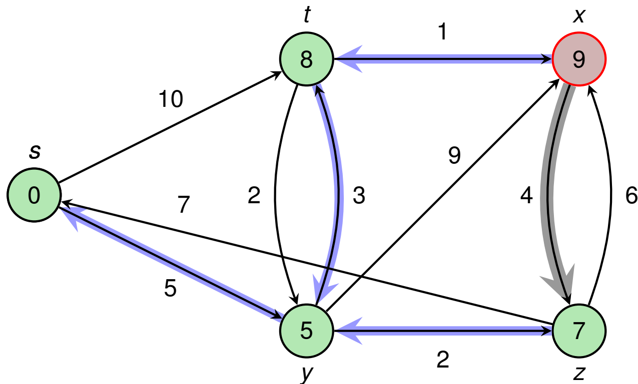

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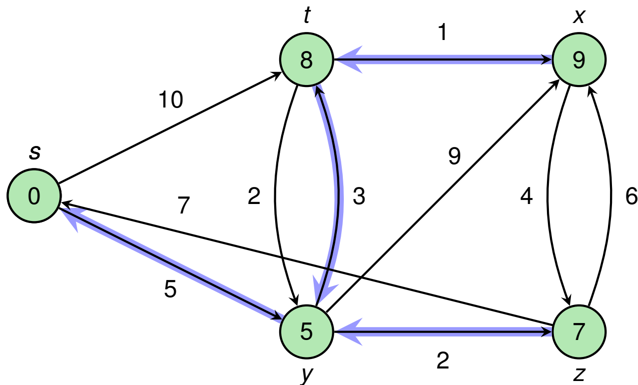

Runtime of Dijkstra's Algorithm

```
DIJKSTRA(G,w,s)
 0: INITIALIZE(G,s)
 1: S = ∅
 2: Q = V
 3: while Q ≠ ∅ do
 4:    u = Extract-Min(Q)
 5:    S = S ∪ {u}
 6:    for each v ∈ G.Adj[u] do
 7:       RELAX(u, v, w)
 8:    end for
 9: end while
```

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Runtime (using Fibonacci Heaps)

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Runtime (using Fibonacci Heaps)

- Initialization (l. 0-2): $\mathcal{O}(V)$

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Runtime (using Fibonacci Heaps)

- Initialization (l. 0-2): $\mathcal{O}(V)$
- ExtractMin (l. 4): $\mathcal{O}(V \cdot \log V)$

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- Initialization (l. 0-2): $\mathcal{O}(V)$
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- DecreaseKey (l. 7): $\mathcal{O}(E \cdot 1)$

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$\Rightarrow$ Overall: $\mathcal{O}(V \log V + E)$

Correctness of Dijkstra's Algorithm

Theorem 24.6

For any directed graph $G = (V, E)$ with non-negative edge weights $w : E \to \mathbb{R}^+$ and source s, Dijkstra terminates with $u.d = u.\delta$ for all $u \in V$.

Proof: Invariant: If v is extracted, $v.d = v.\delta$

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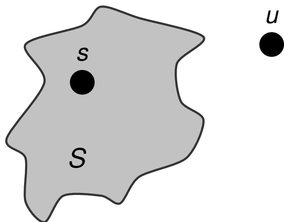
- Let u be the first vertex with this property



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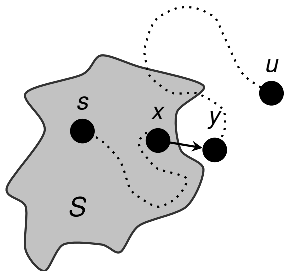

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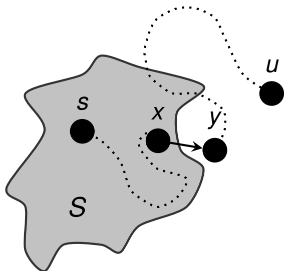
$\Rightarrow$

$$u.d$$



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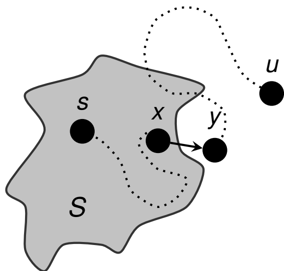
$\Rightarrow$

$$u.d \leq y.d$$



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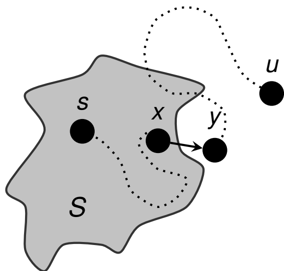
u is extracted before y



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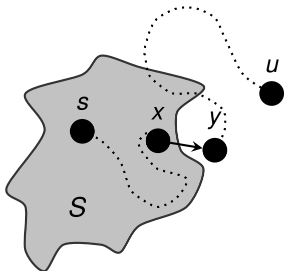

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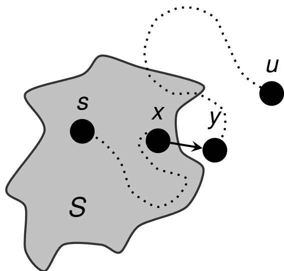
$$u.d \leq y.d = y.\delta$$

since $x.d = x.\delta$ when x is extracted, and then (x, y) is relaxed $\Rightarrow$ Convergence Property



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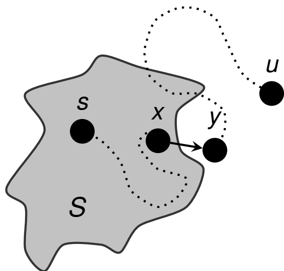
$\Rightarrow$

$$u.\delta < u.d \leq y.d = y.\delta$$



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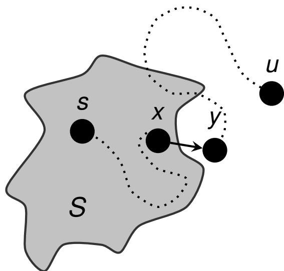

This contradicts that y is on a shortest path from s to u. $\square$

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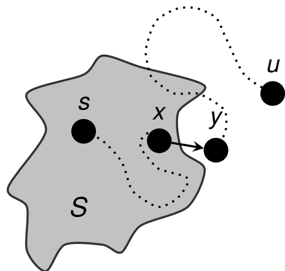

This contradicts that y is on a shortest path from s to u.

This step requires non-negative weights!

Correctness of Dijkstra's Algorithm

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For any directed graph $G = (V, E)$ with non-negative edge weights $w : E \to \mathbb{R}^+$ and source s, Dijkstra terminates with $u.d = u.\delta$ for all $u \in V$.

There are edge cases like $s = x$ and/or $y = u$!

Proof: Invariant: If v is extracted, $v.d = v.\delta$

- Suppose there is $u \in V$, when extracted,

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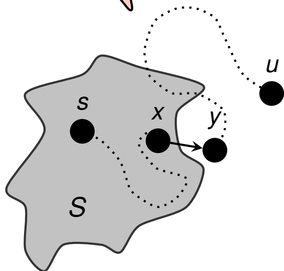

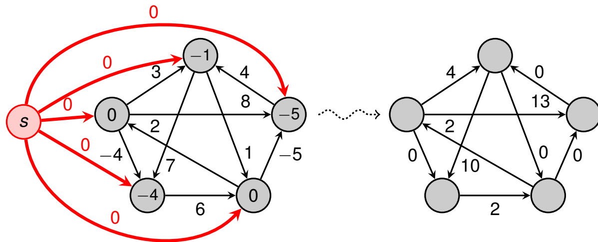

6.5: All-Pairs Shortest Paths

Frank Stajano

Thomas Sauerwald

Lent 2015

UNIVERSITY OF CAMBRIDGE

Outline

Bellman-Ford

Dijkstra's Algorithm

All-Pairs Shortest Path

APSP via Matrix Multiplication

Johnson's Algorithm

Formalizing the Problem

All-Pairs Shortest Path Problem

- Given: directed graph $G = (V, E)$, $V = \{1, 2, \ldots, n\}$, with edge weights represented by a matrix W:

$$w_{i,j} = \begin{cases} \text{weight of edge } (i,j) & \text{for an edge } (i,j) \in E, \\ \infty & \text{if there is no edge from } i \text{ to } j, \\ 0 & \text{if } i = j. \end{cases}$$

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- Goal: Obtain a matrix of shortest path weights L, that is

$$\ell_{i,j} = \begin{cases} \text{weight of a shortest path from } i \text{ to } j, & \text{if } j \text{ is reachable from } i \\ \infty & \text{otherwise.} \end{cases}$$

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Here we will only compute the weight of the shortest path without keeping track of the edges of the path!

Outline

Bellman-Ford

Dijkstra's Algorithm

All-Pairs Shortest Path

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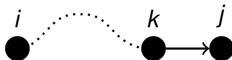
Johnson's Algorithm

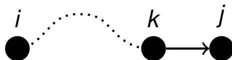
A Recursive Approach



Basic Idea

- Any shortest path from i to j of length $k \geq 2$ is the concatenation of a shortest path of length $k - 1$ and an edge

A Recursive Approach



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- Let $\ell_{i,j}^{(m)}$ be min. weight of any path from i to j with at most m edges

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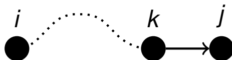

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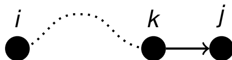
A Recursive Approach



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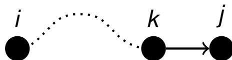
A Recursive Approach



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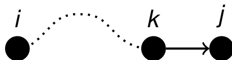
A Recursive Approach



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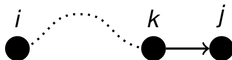
A Recursive Approach



Basic Idea

- Any shortest path from i to j of length $k \geq 2$ is the concatenation of a shortest path of length $k-1$ and an edge

- Let $\ell_{i,j}^{(m)}$ be min. weight of any path from i to j with at most m edges
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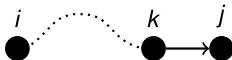

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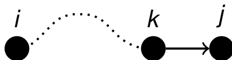

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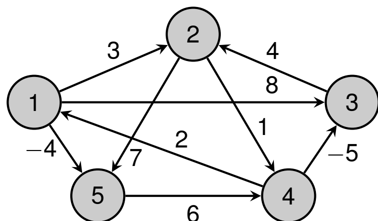
Example of Shortest Path via Matrix Multiplication (Figure 25.1)



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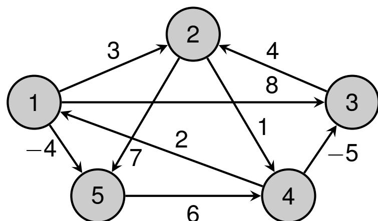

$$L^{(1)} = W = \begin{pmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix}$$

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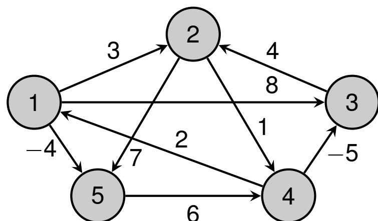

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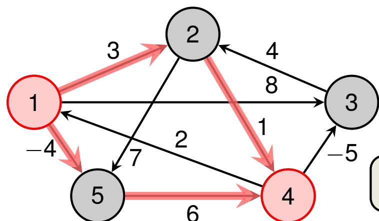

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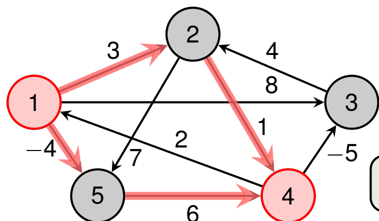

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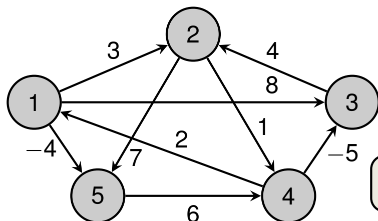

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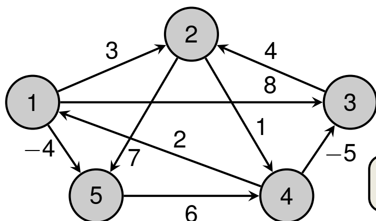

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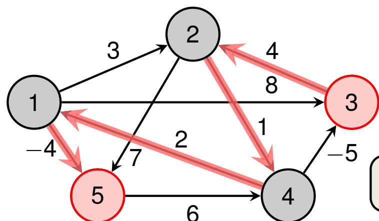

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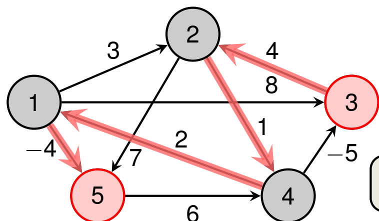

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$L^{(m)}$ can be computed in $\mathcal{O}(n^3)$

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We need $L^{(4)} = L^{(2)} \cdot L^{(2)} = L^{(3)} \cdot L^{(1)}$! (see Ex. 25.1-4)

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Outline

Bellman-Ford

Dijkstra's Algorithm

All-Pairs Shortest Path

APSP via Matrix Multiplication

Johnson's Algorithm

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Overview

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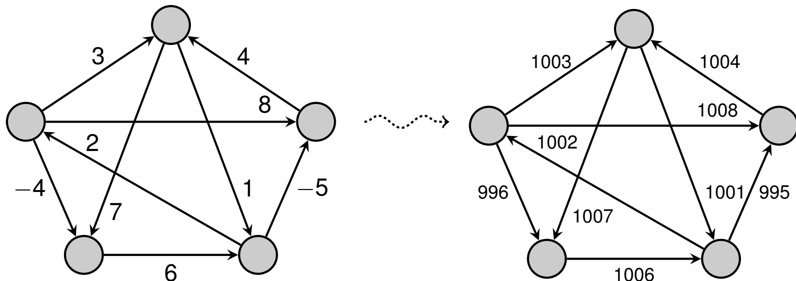
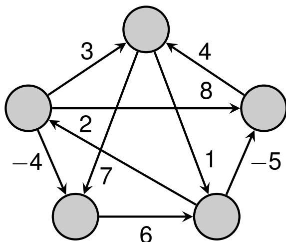


How Johnson's Algorithm works

Johnson's Algorithm

1. Add a new vertex s and directed edges (s, v), $v \in V$, with weight 0



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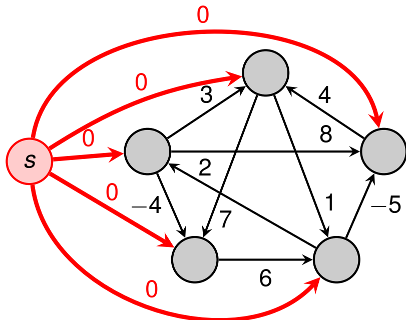

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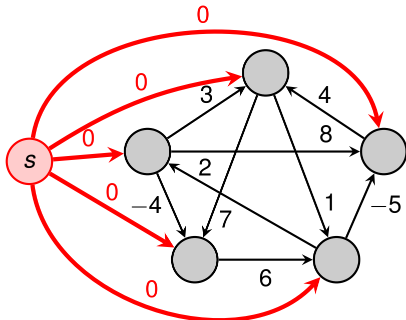
2. Run Bellman-Ford on this augmented graph with source s



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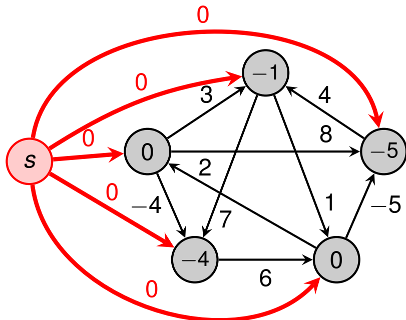

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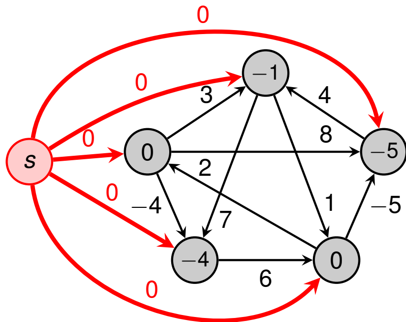

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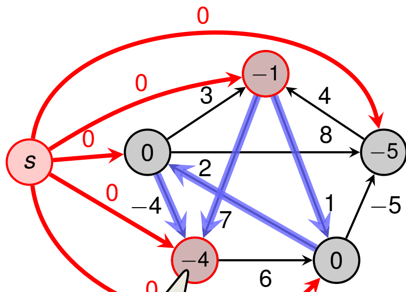
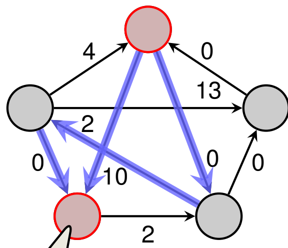
Runtime: $O(V \cdot E + V \cdot (V \log V + E))$





Correctness of Johnson's Algorithm

Theorem

For any graph $G = (V, E, w)$ without negative-weight cycles:

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$$\Rightarrow \quad \widetilde{w}(u,v) \geq 0$$

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Comparison of all Shortest-Path Algorithms

Algorithm	SSSP		APSP		negative weights
	sparse	dense	sparse	dense	
Bellman-Ford	V^2	V^3	V^3	V^4	✓
Dijkstra	$V \log V$	V^2	$V^2 \log V$	V^3	X
Matrix Mult.	–	–	$V^3 \log V$	$V^3 \log V$	(✓)
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can handle negative weight edges, but not negative-weight cycles