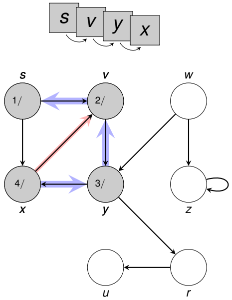


6.1 & 6.2: Graph Searching

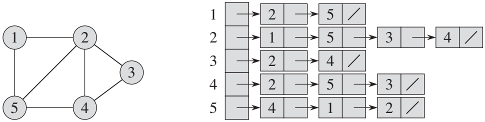
Frank Stajano

Thomas Sauerwald

Lent 2015

UNIVERSITY OF CAMBRIDGE

Representations of Directed and Undirected Graphs



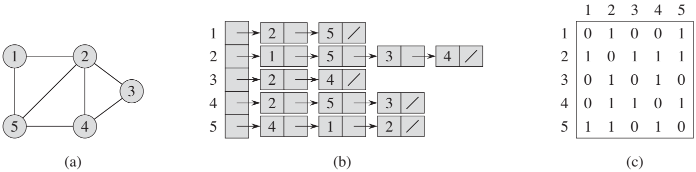
(a) (b)

	1	2	3	4	5
1	0	1	0	0	1
2	1	0	1	1	1
3	0	1	0	1	0
4	0	1	1	0	1
5	1	1	0	1	0

(c)

Figure 22.1 Two representations of an undirected graph. **(a)** An undirected graph G with 5 vertices and 7 edges. **(b)** An adjacency-list representation of G. **(c)** The adjacency-matrix representation of G.

Representations of Directed and Undirected Graphs



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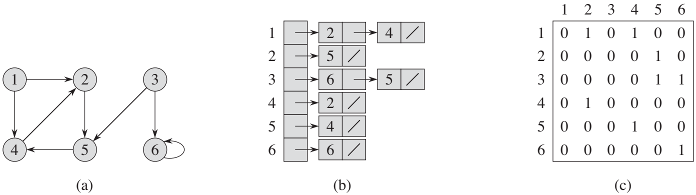
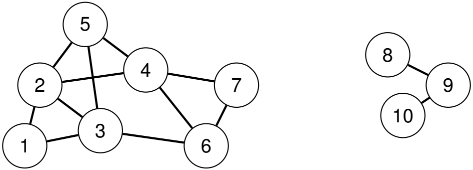


Figure 22.2 Two representations of a directed graph. **(a)** A directed graph G with 6 vertices and 8 edges. **(b)** An adjacency-list representation of G. **(c)** The adjacency-matrix representation of G.

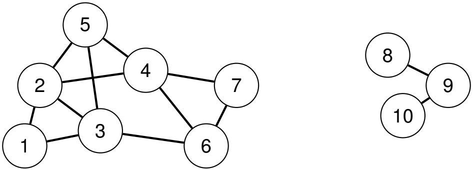
Graph Searching



Overview

- Graph searching means traversing a graph via the edges in order to visit all vertices
- useful for identifying connected components, computing the diameter etc.

Graph Searching



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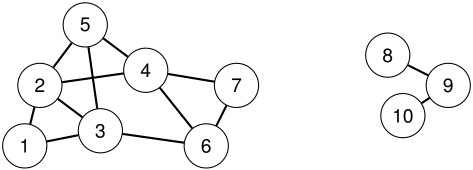


Overview

- Graph searching means traversing a graph via the edges in order to visit all vertices
- useful for identifying connected components, computing the diameter etc.
- Two strategies: Breadth-First-Search and Depth-First-Search

Measure time complexity in terms of the size of V and E (often write just V instead of $|V|$, and E instead of $|E|$)

Outline

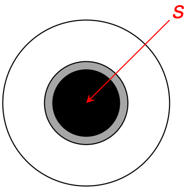
Breadth-First Search

Depth-First Search

Topological Sort

Minimum Spanning Tree Problem

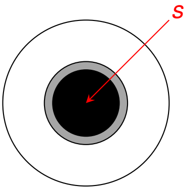
Breadth-First Search: Basic Ideas



Basic Idea

- Given an undirected/directed graph $G = (V, E)$ and source vertex s

Breadth-First Search: Basic Ideas



Basic Idea

- Given an undirected/directed graph $G = (V, E)$ and source vertex s
- BFS sends out a wave from $s \rightsquigarrow$ compute distances/shortest paths

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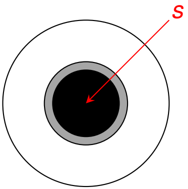

Basic Idea

- Given an undirected/directed graph $G = (V, E)$ and source vertex s
- BFS sends out a wave from $s \rightsquigarrow$ compute distances/shortest paths
- Vertex Colours:

White = Unvisited

Grey = Visited, but not all neighbors (=adjacent vertices)

Black = Visited and all neighbors

Breadth-First-Search: Pseudocode

```
def bfs(G,s)
    Run BFS on the given graph G
    starting from source s
   assert(s in G.vertices())
# Initialize graph and queue
for v in G.vertices():
      v.predecessor = None
      v.d = Infinity # .d = distance from s
      v.colour = "white"
Q = Queue()
# Visit source vertex
s.d = 0
s.colour = "grey"
Q.insert(s)
# Visit the adjacents of each vertex in Q
while not Q.isEmpty():
    u = Q.extract()
    assert (u.colour == "grey")
    for v in u.adjacent()
        if v.colour = "white"
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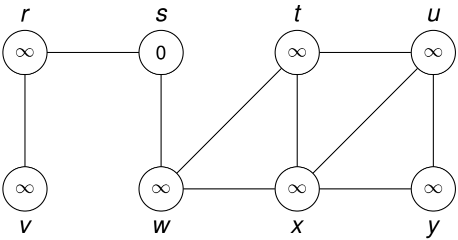
Assuming that all executions of the FOR-loop for u takes $O(|u.adj|)$ (**adjacency list model!**)

$$\sum_{u \in V} |u.adj| = 2|E|$$

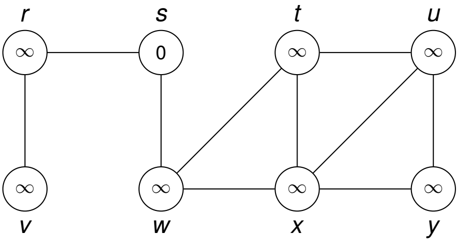
Complete Execution of BFS (Figure 22.3)

Queue:



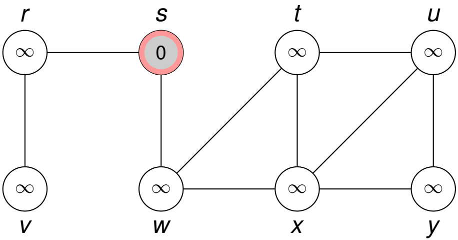
Complete Execution of BFS (Figure 22.3)

Queue: *s*



Complete Execution of BFS (Figure 22.3)

Queue: ~~s~~



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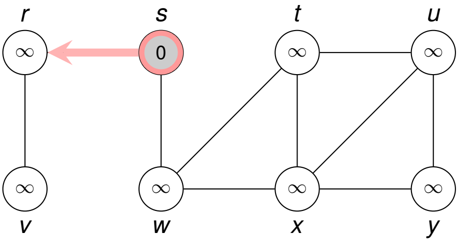
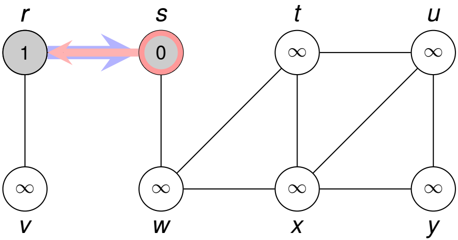
Queue: ~~*s*~~



Complete Execution of BFS (Figure 22.3)

Queue: ~~s~~ r



Complete Execution of BFS (Figure 22.3)

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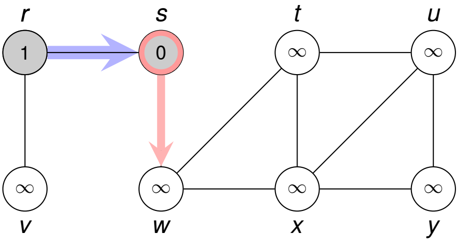

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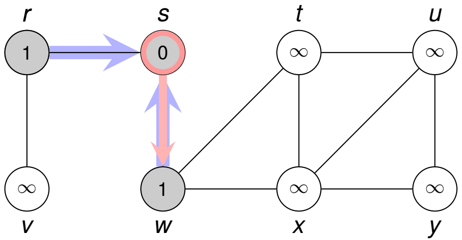
Queue: ~~s~~ r w



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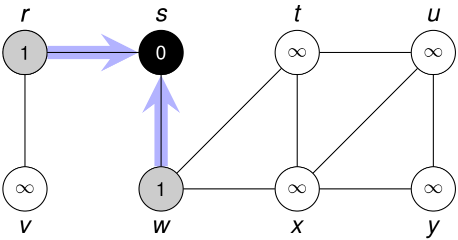

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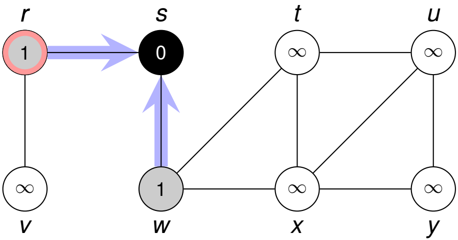
Queue: ~~*s*~~ ~~*r*~~ *w*



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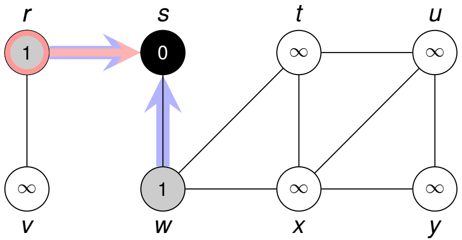

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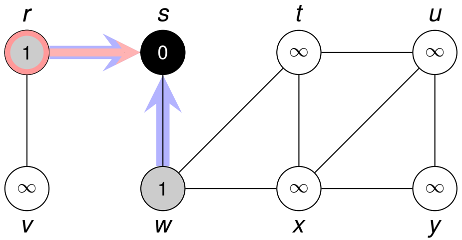

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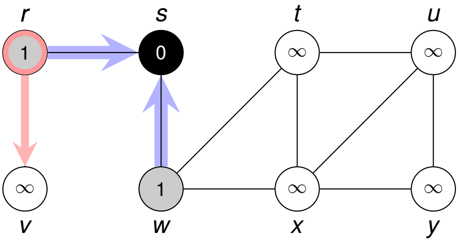
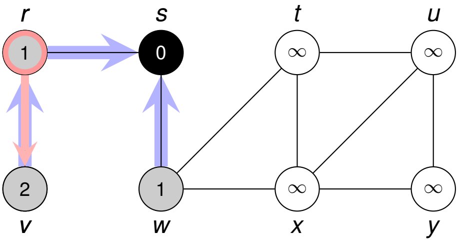


Complete Execution of BFS (Figure 22.3)

Queue: ~~*s*~~ ~~*r*~~ *w* *v*



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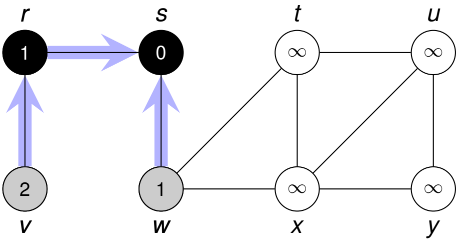

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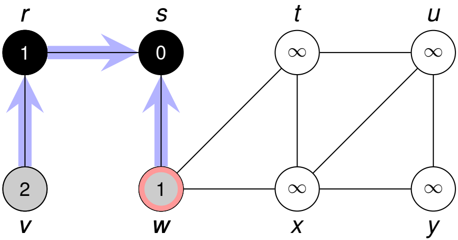

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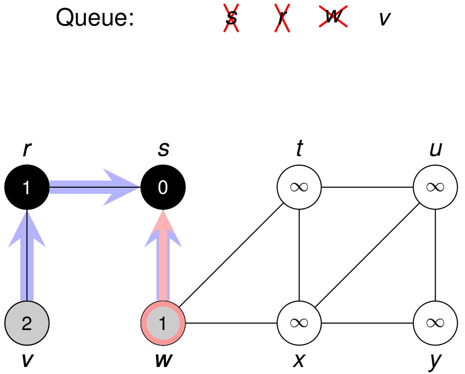

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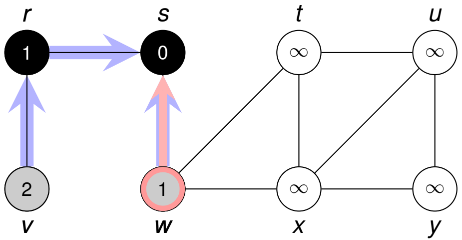

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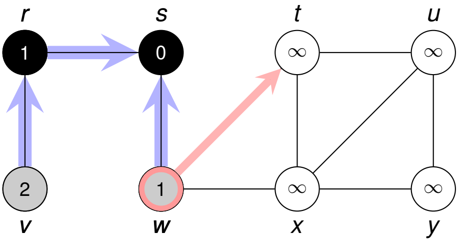

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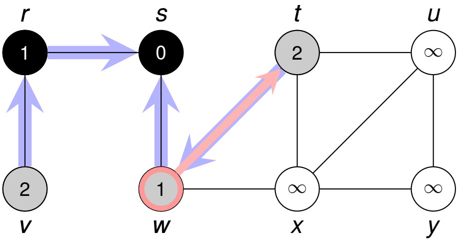
Queue: ~~s~~ ~~r~~ ~~w~~ *v* *t*



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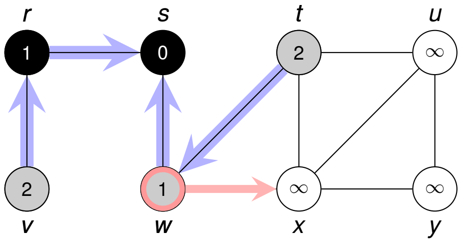

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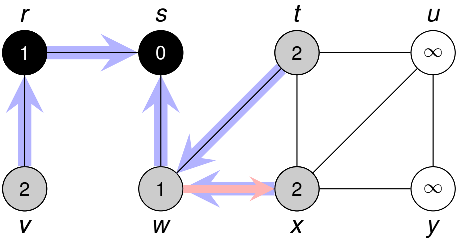

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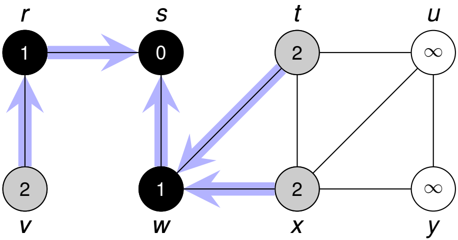

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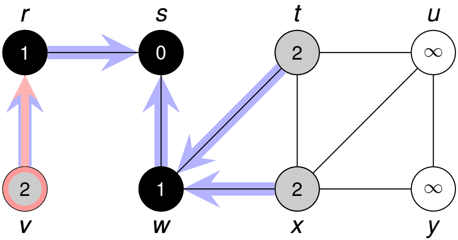

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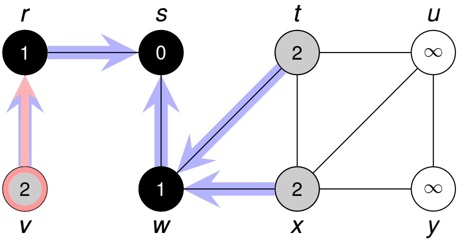

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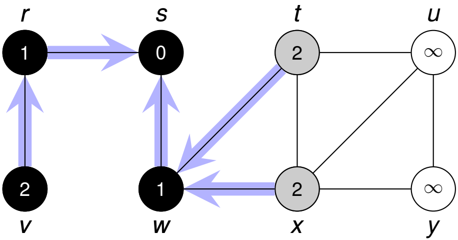

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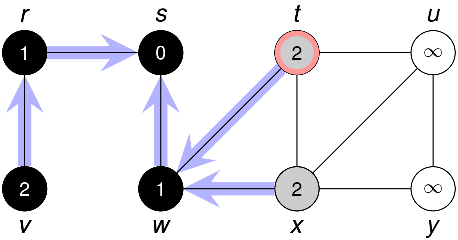

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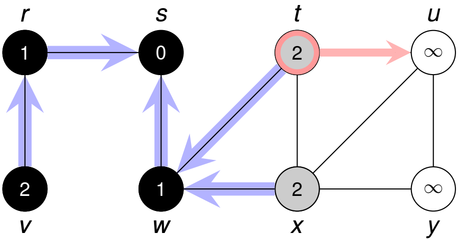

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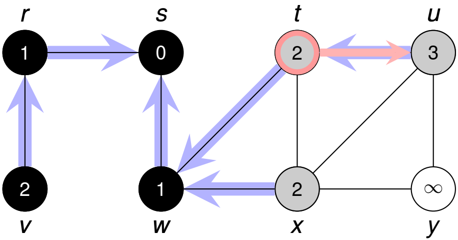

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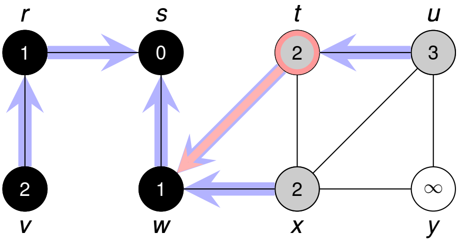

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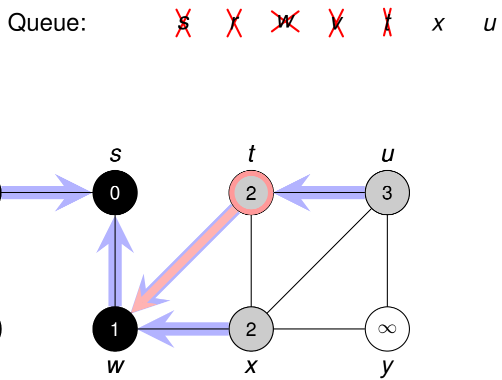

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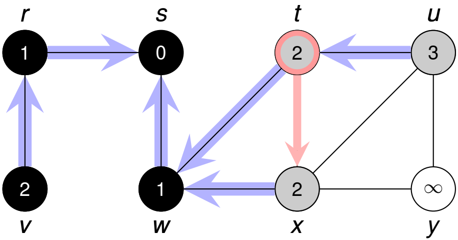

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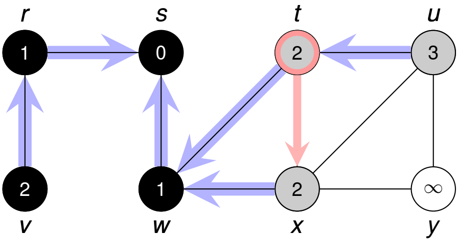

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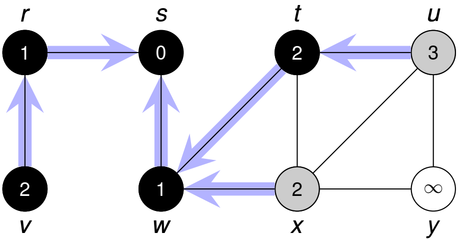

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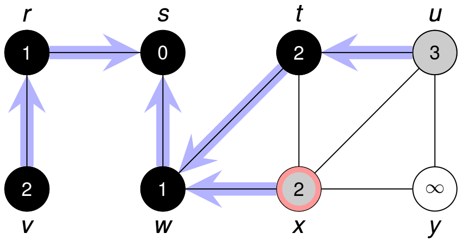

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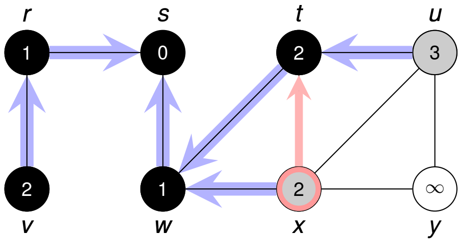

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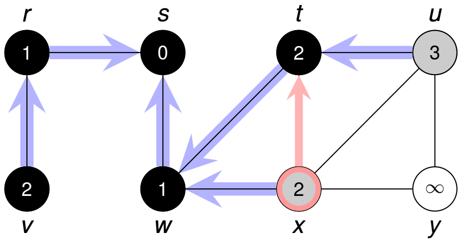

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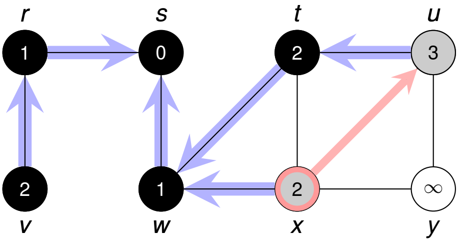

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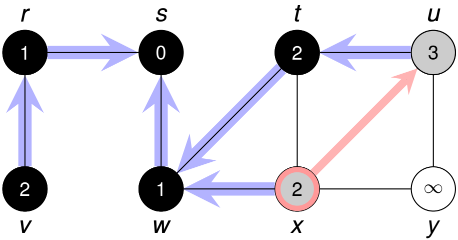

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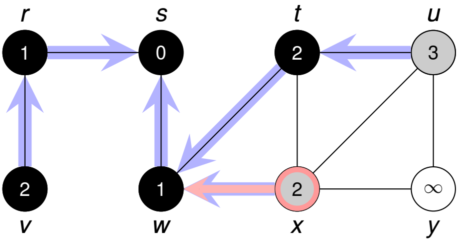

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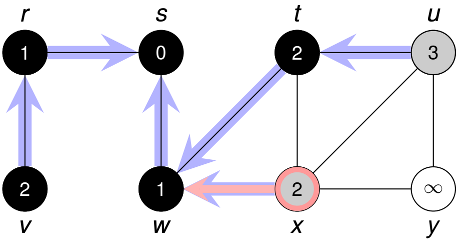

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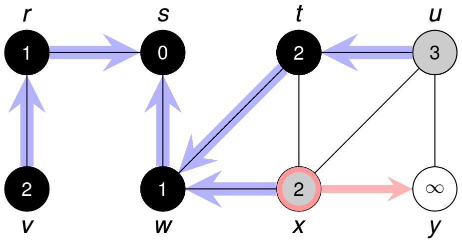

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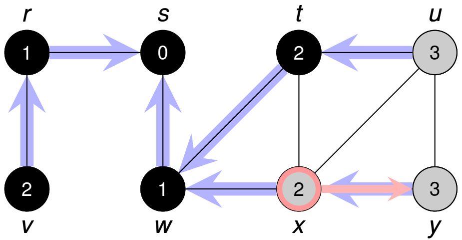

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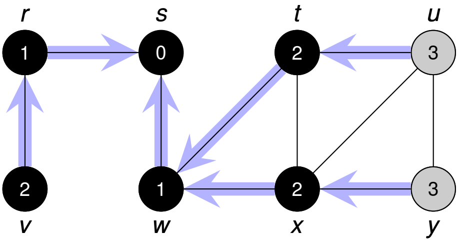

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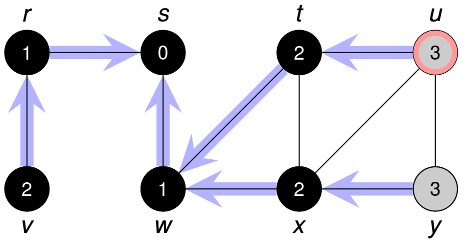

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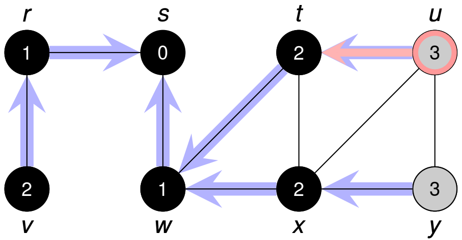

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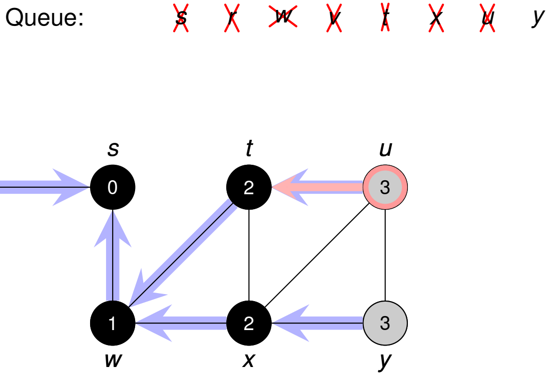

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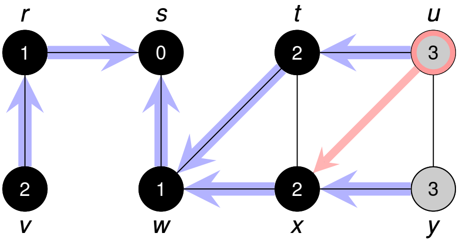

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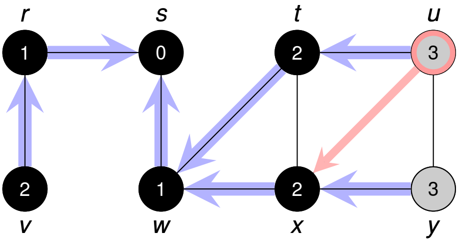

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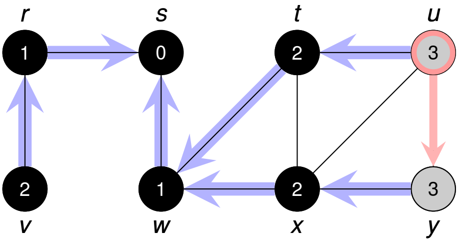

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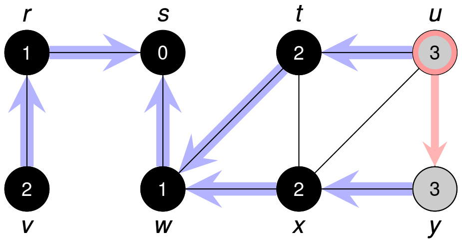

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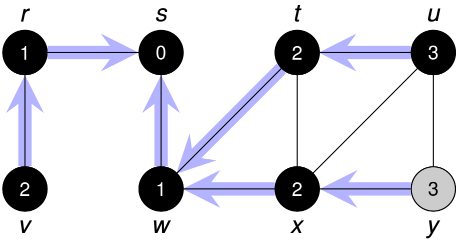

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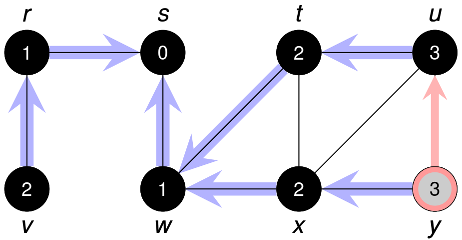

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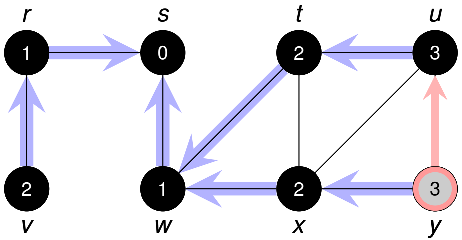

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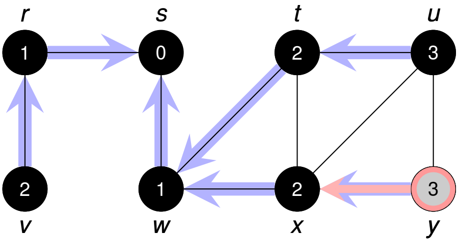

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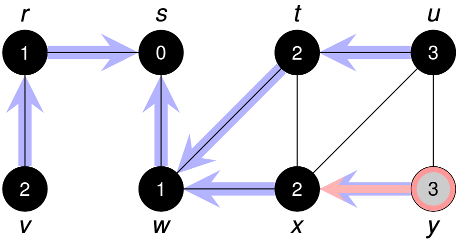

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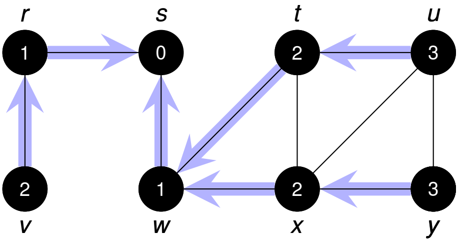


Outline

Breadth-First Search

Depth-First Search

Topological Sort

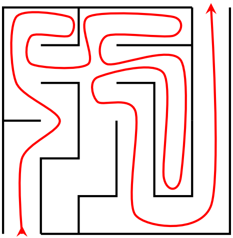
Minimum Spanning Tree Problem

Depth-First Search: Basic Ideas

Basic Idea

- Given an undirected/directed graph $G = (V, E)$ and source vertex s

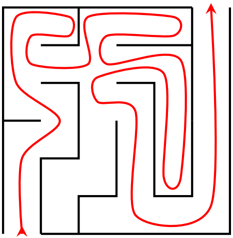
Depth-First Search: Basic Ideas



Basic Idea

- Given an undirected/directed graph $G = (V, E)$ and source vertex s
- As soon as we discover a vertex, explore from it $\rightsquigarrow$ Solving Mazes

Depth-First Search: Basic Ideas



Basic Idea

- Given an undirected/directed graph $G = (V, E)$ and source vertex s
- As soon as we discover a vertex, explore from it $\rightsquigarrow$ Solving Mazes
- Two time stamps for every vertex: Discovery Time, Finishing Time

Depth-First-Search: Pseudocode

```
def dfs(G,s):
    Run DFS on the given graph G
    starting from the given source s

  assert(s in G.vertices())

# Initialize graph
  for v in G.vertices():
      v.predecessor = None
      v.colour = "white"
  dfsRecurse(G,s)
```

```
def dfsRecurse(G,s):
  s.colour = "grey"
  s.d = time() # .d = discovery time
  for v in s.adjacent()
      if v.colour = "white"
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  s.colour = "black"
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- Discovery and Finish times, *.d* and *.f*
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  White = Unvisited

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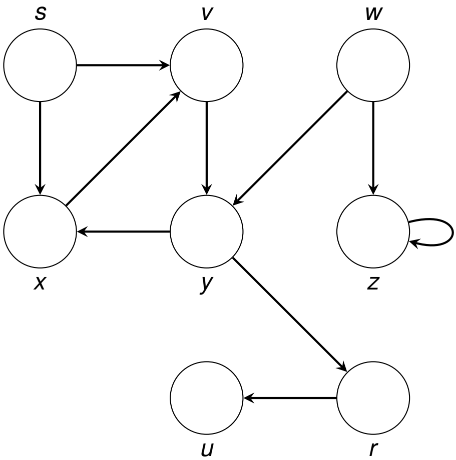
  Black = Visited and all neighbors
- Runtime $O(V + E)$

Complete Execution of DFS



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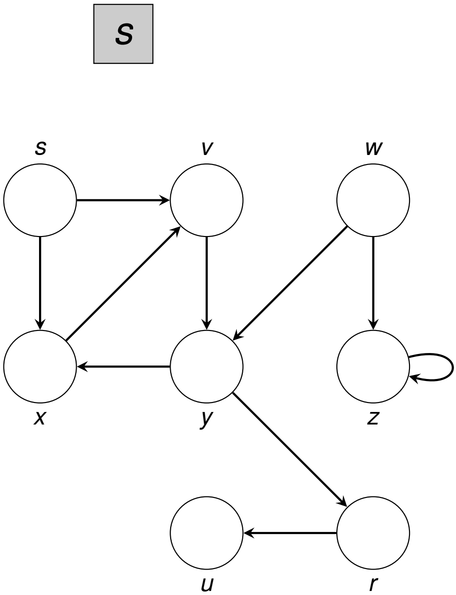

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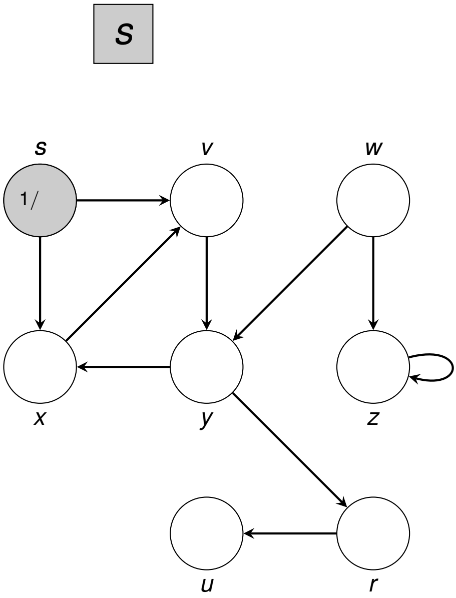

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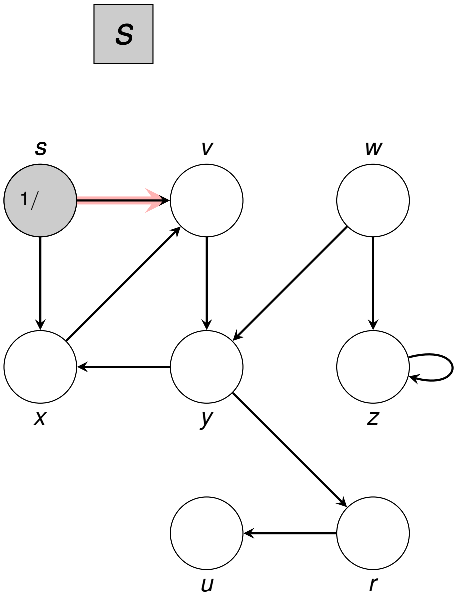

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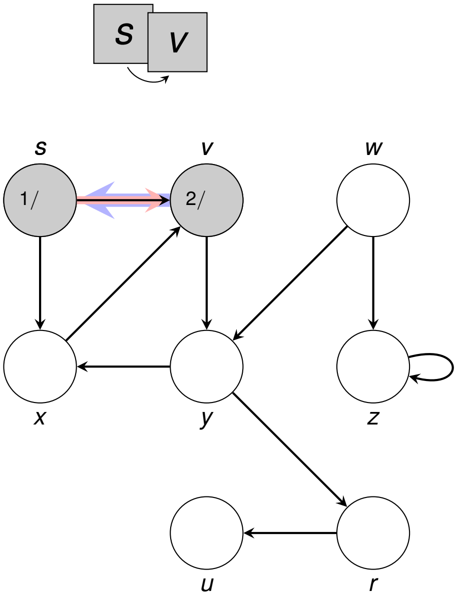

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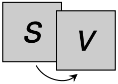

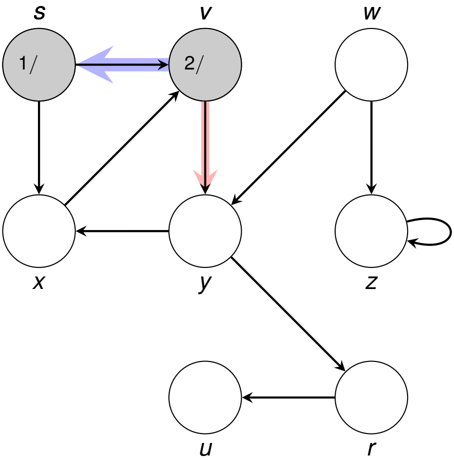

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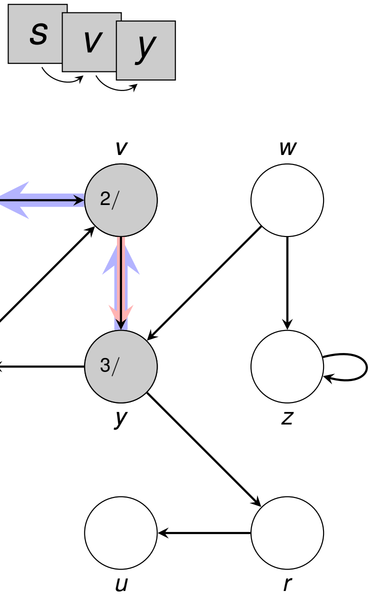

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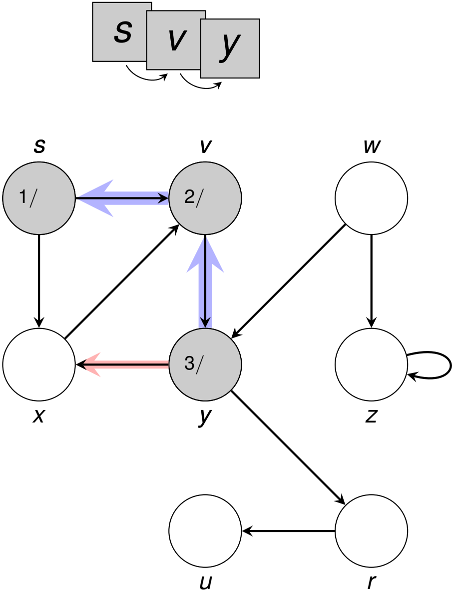

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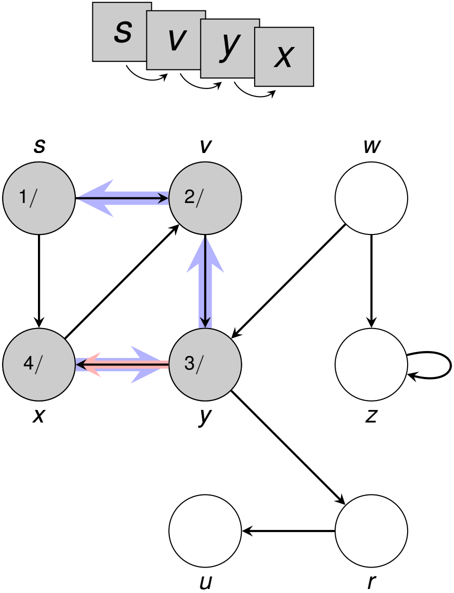

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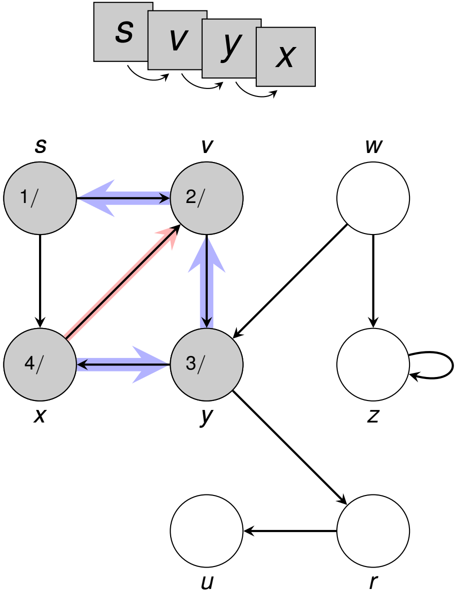

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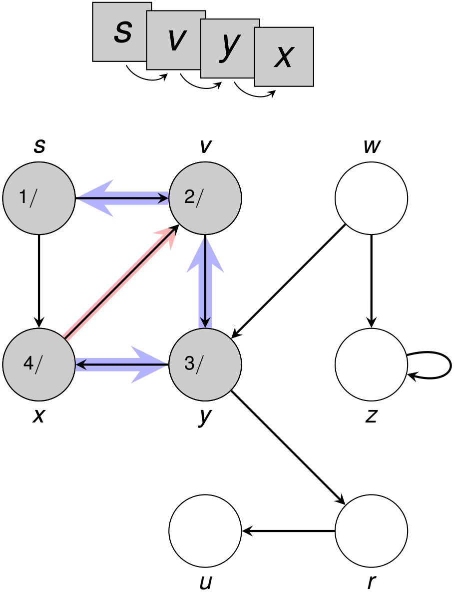

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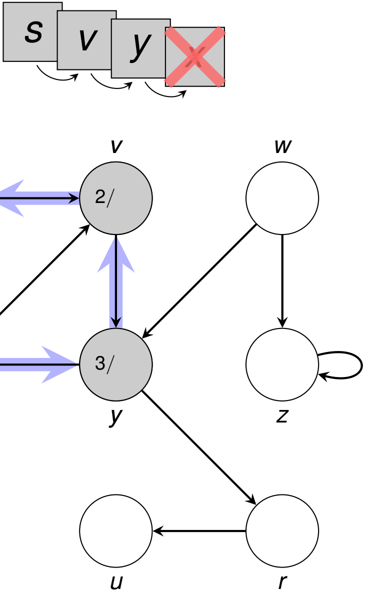

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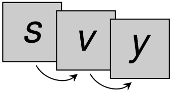
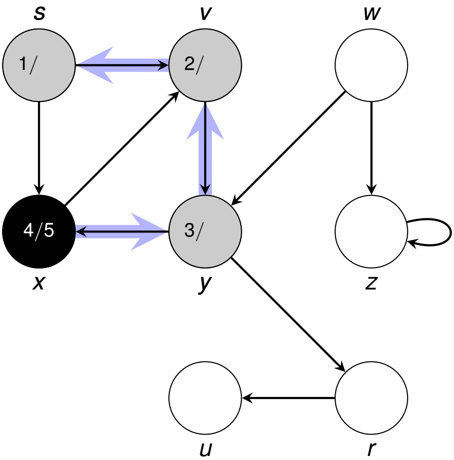
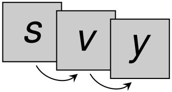
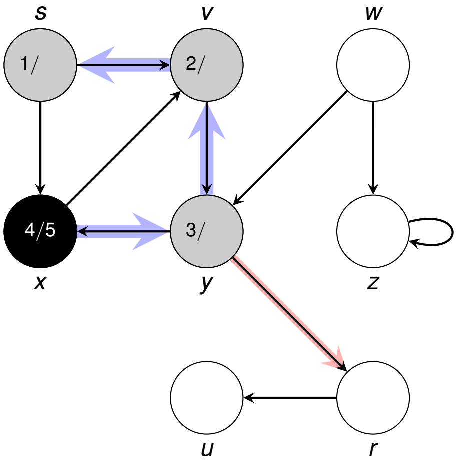


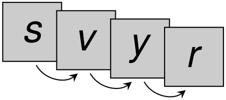


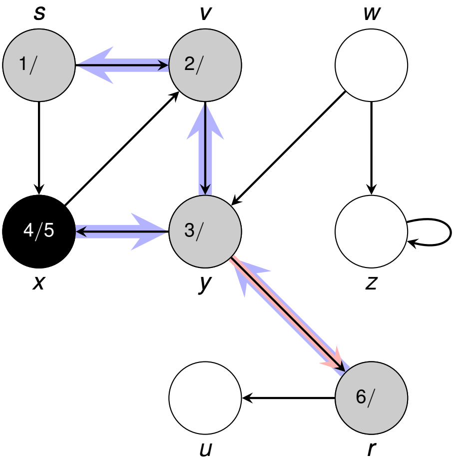
Complete Execution of DFS



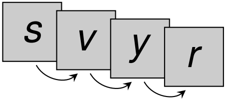


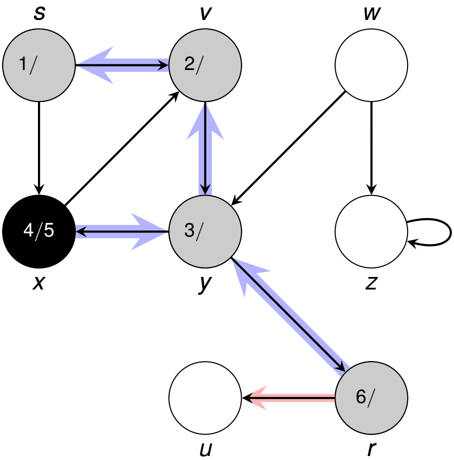
Complete Execution of DFS





Complete Execution of DFS





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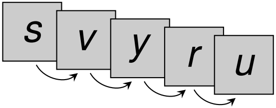

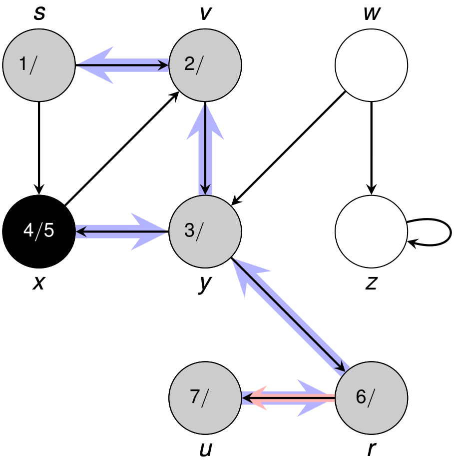

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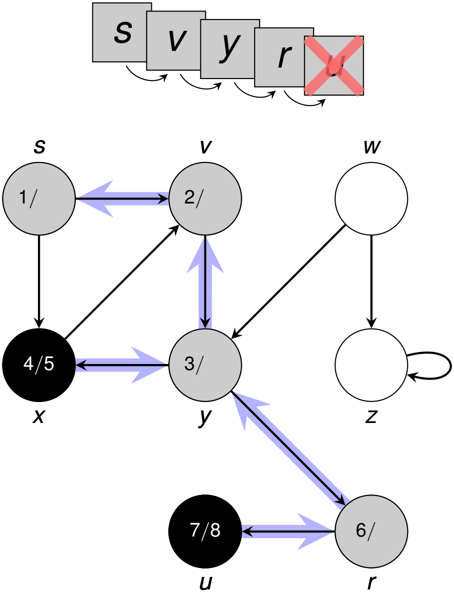

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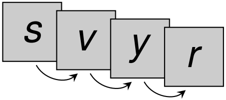

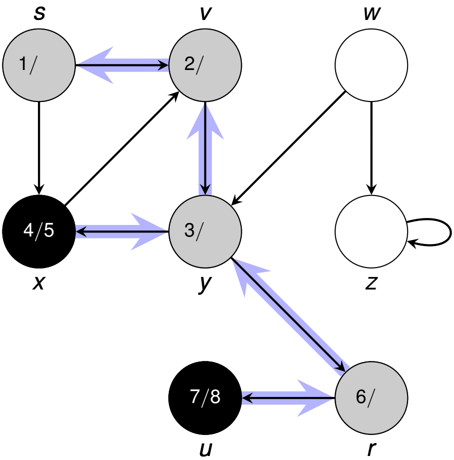

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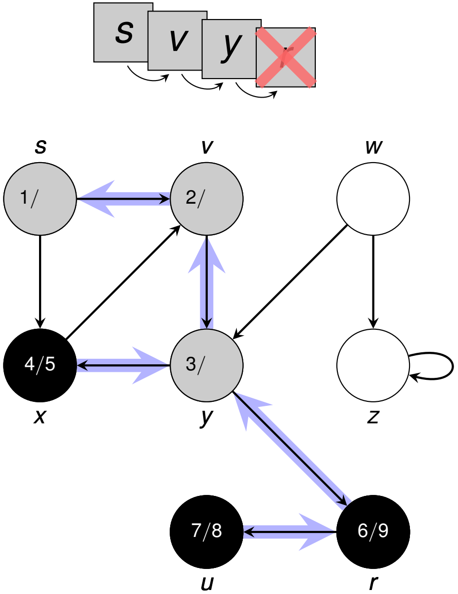

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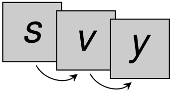

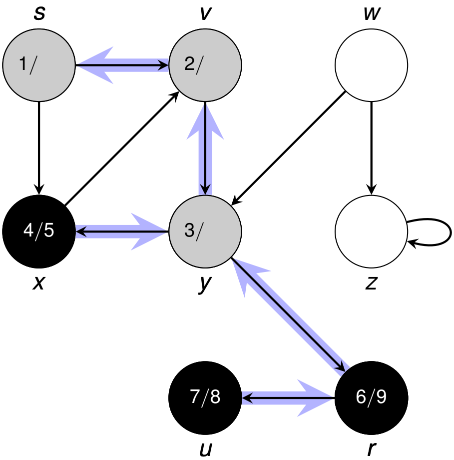

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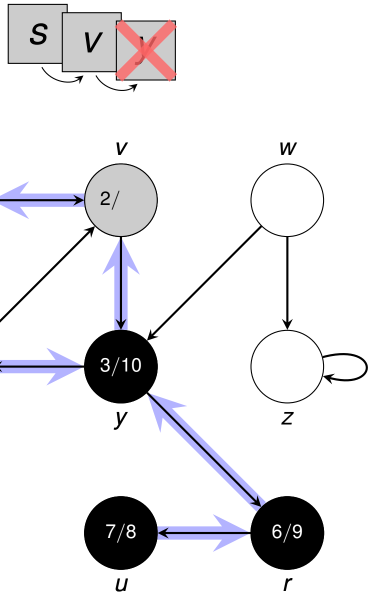

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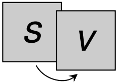

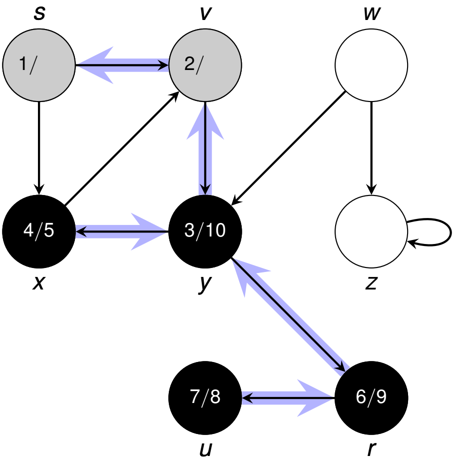

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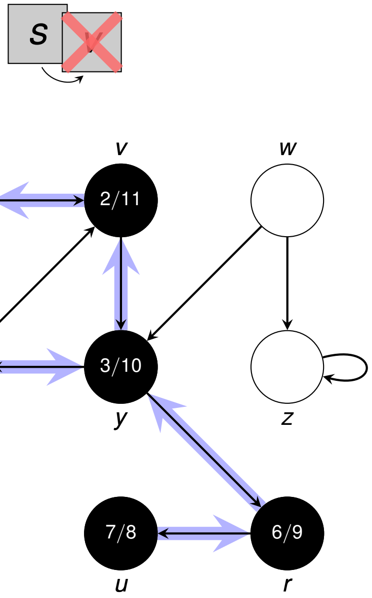

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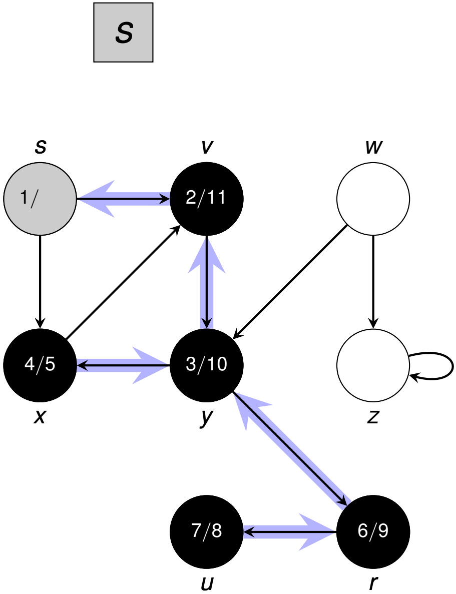

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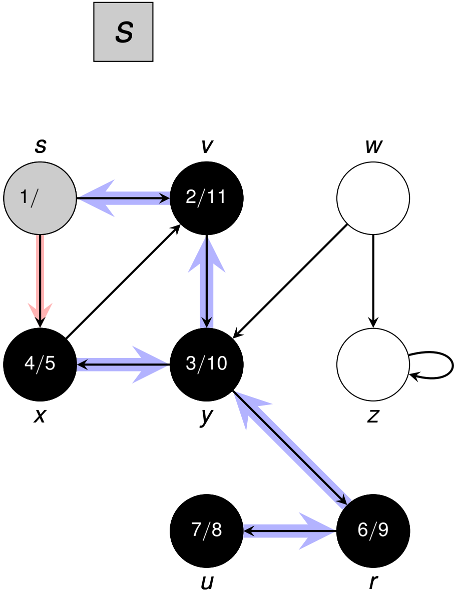

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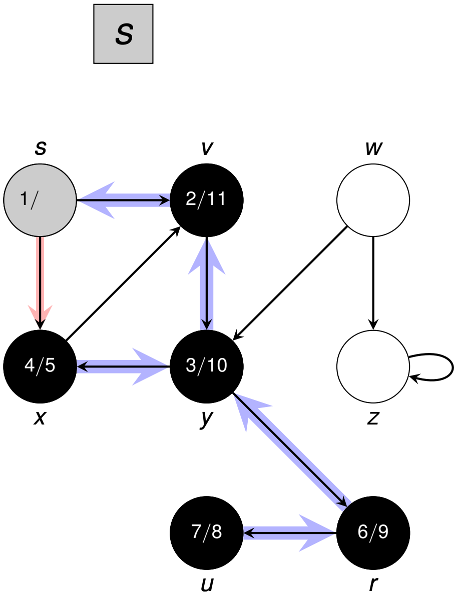

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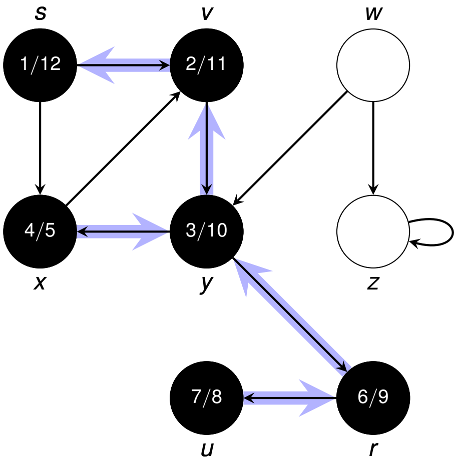

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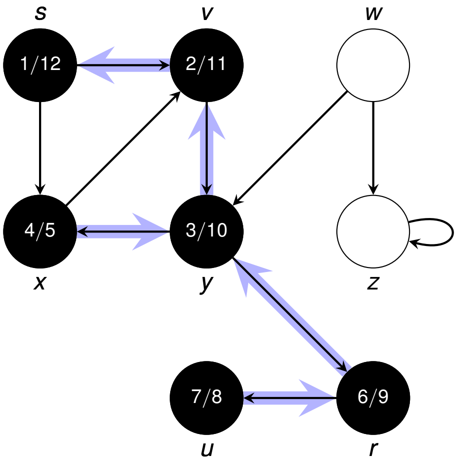

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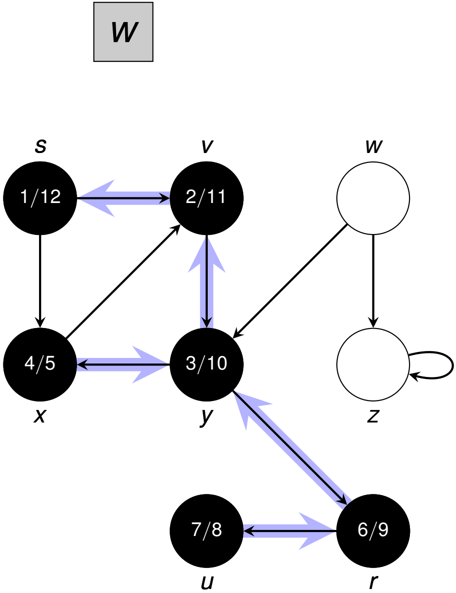

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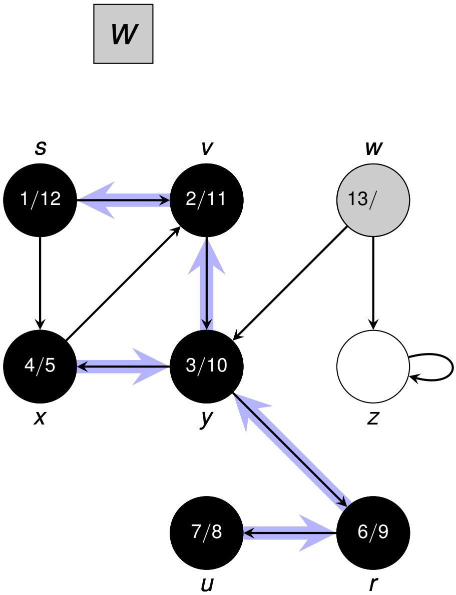

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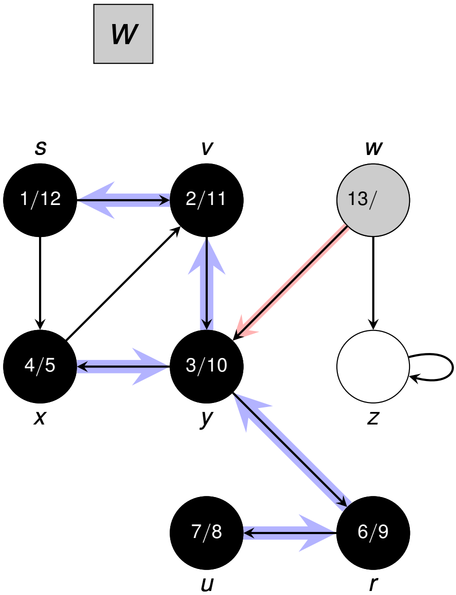

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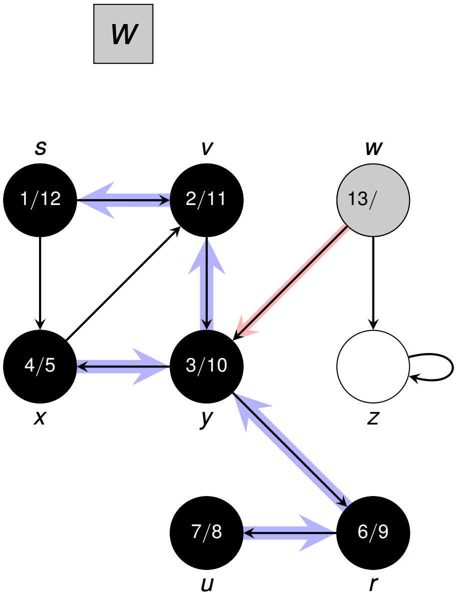

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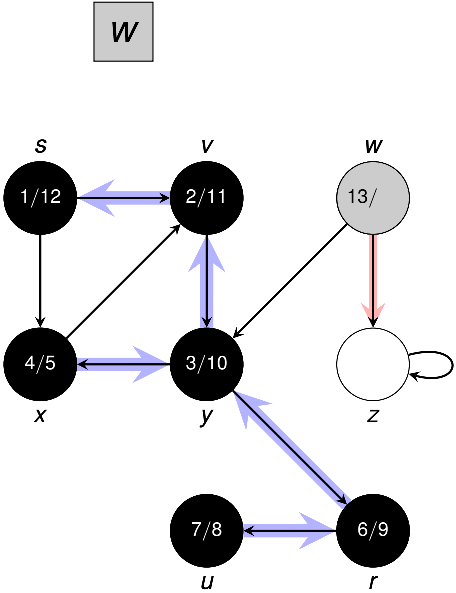

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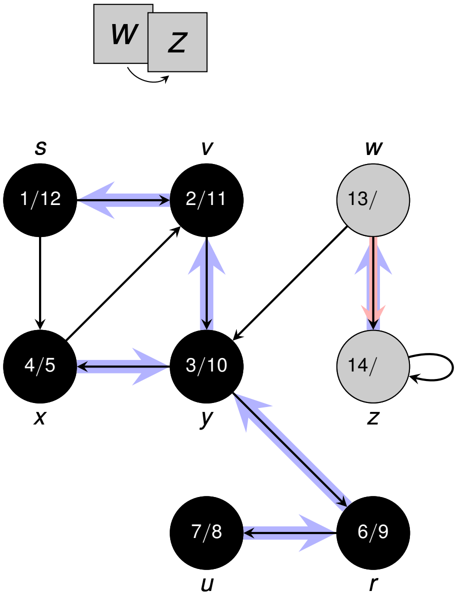

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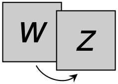

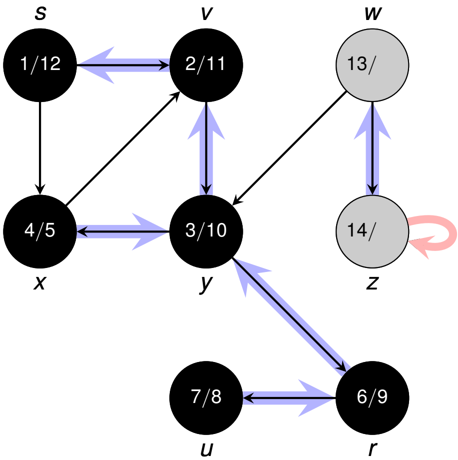

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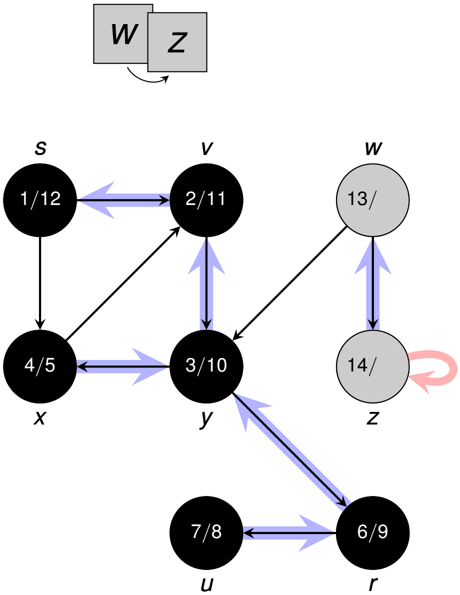

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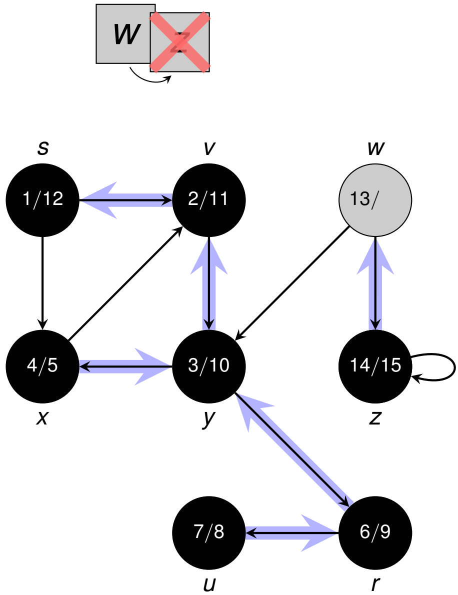

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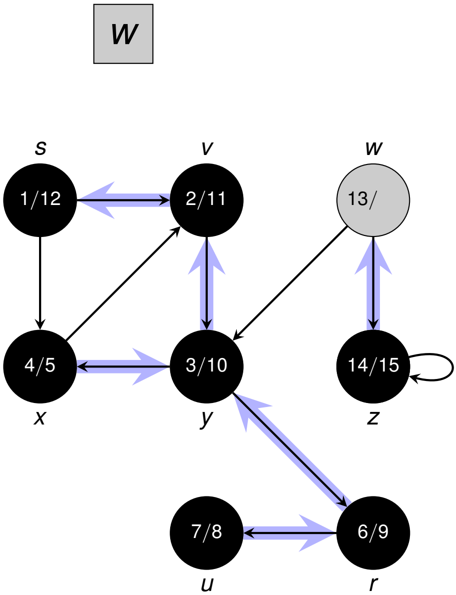

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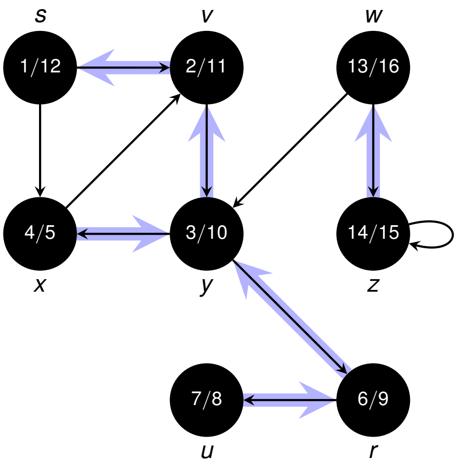

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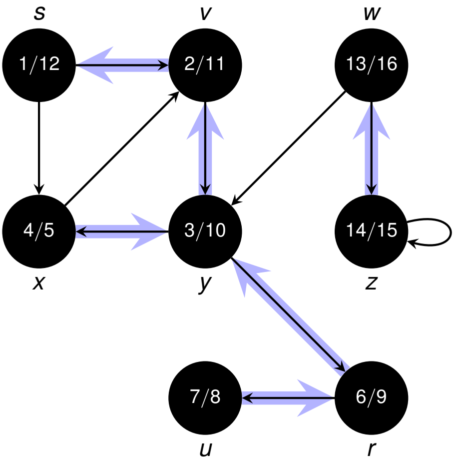
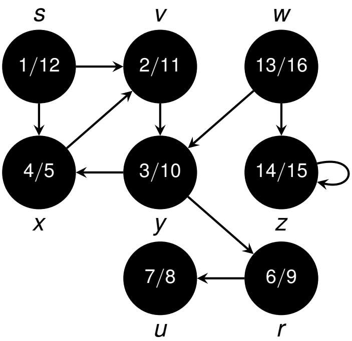
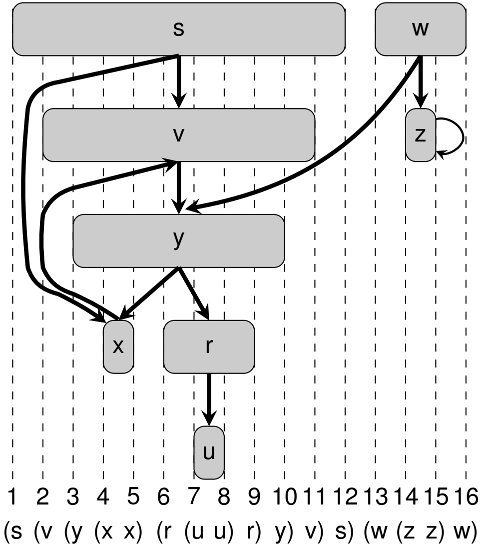


Paranthesis Theorem (Theorem 22.7)





Outline

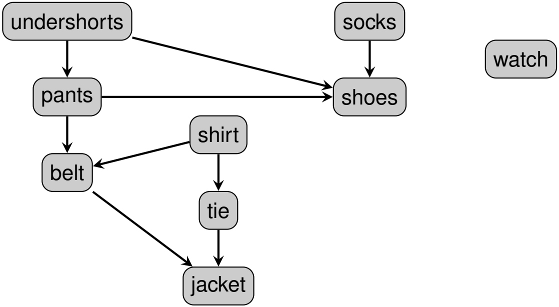
Breadth-First Search

Depth-First Search

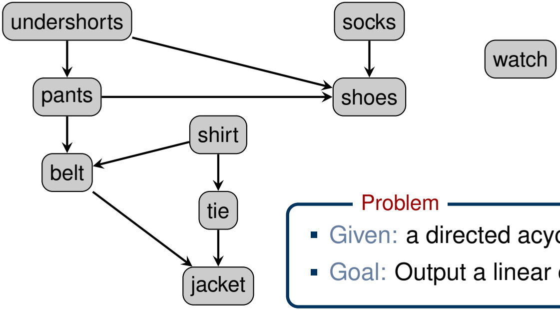
Topological Sort

Minimum Spanning Tree Problem

Topological Sort



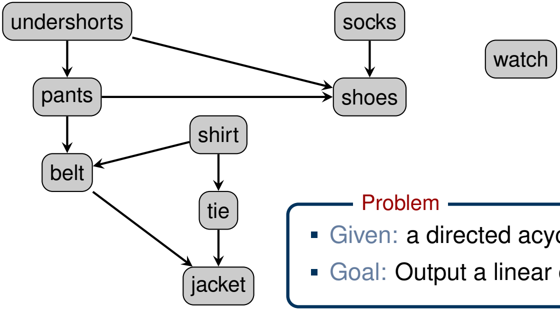
Topological Sort



Problem

- Given: a directed acyclic graph (DAG)
- Goal: Output a linear ordering of all vertices

Topological Sort



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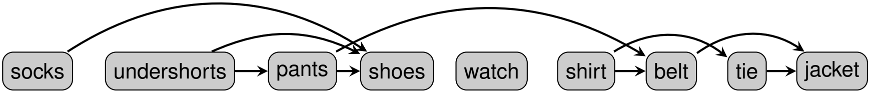
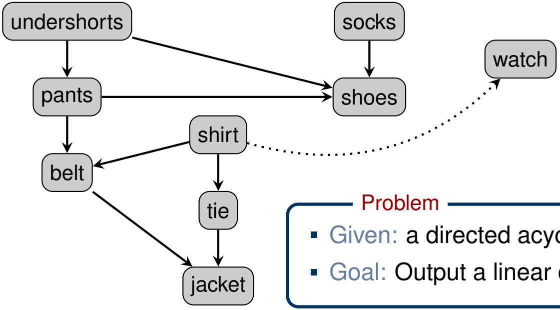


Topological Sort



Problem

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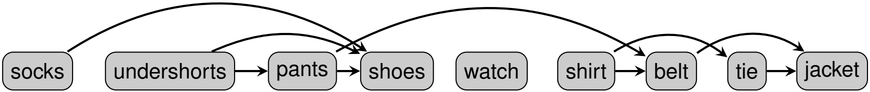
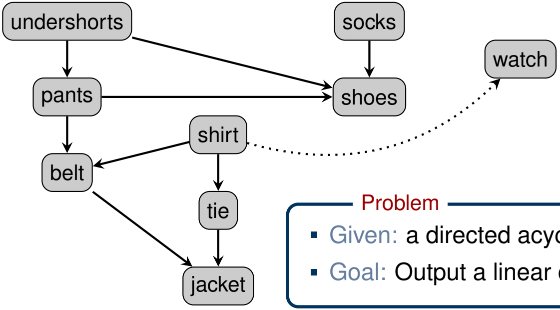


Topological Sort



Problem

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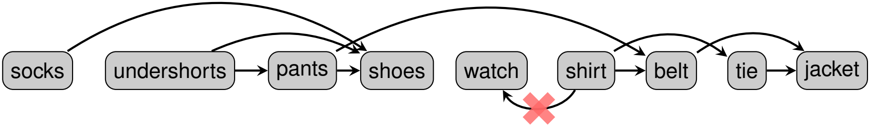
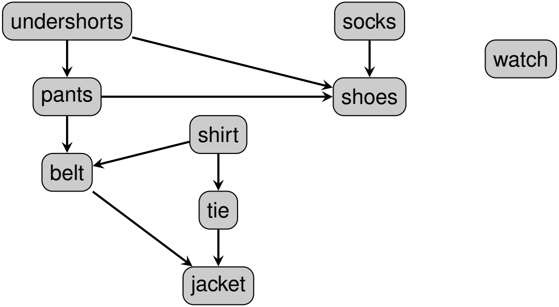


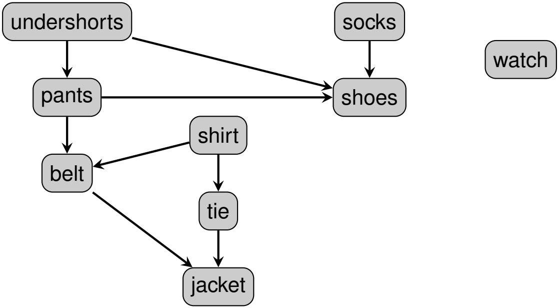
Solving Topological Sort



Knuth's Algorithm (1968)

- Perform DFS's so that all vertices are visited
- Output vertices in decreasing order of their finishing time

Solving Topological Sort



Knuth's Algorithm (1968)

- Perform DFS's so that all vertices are visited
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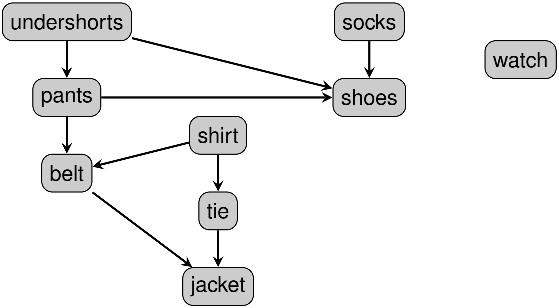
Runtime $O(V + E)$

Solving Topological Sort



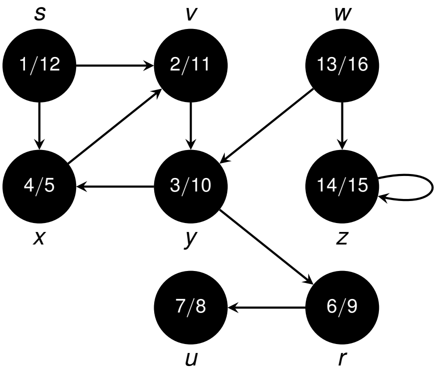
Knuth's Algorithm (1968)

- Perform DFS's so that all vertices are visited
- Output vertices in decreasing order of their finishing time

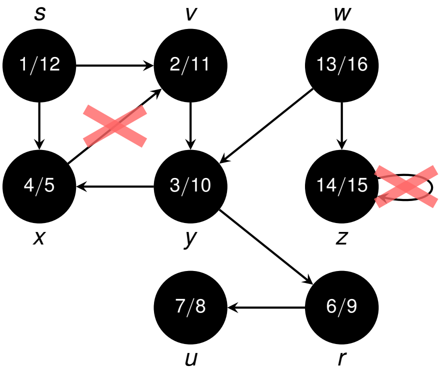
Runtime $O(V + E)$

Don't need to sort the vertices – use DFS directly!

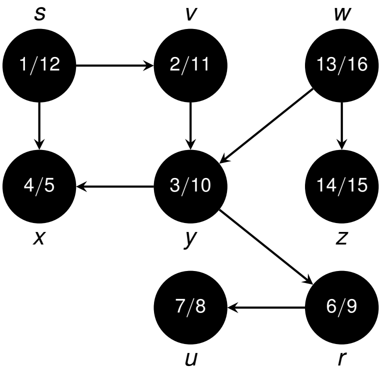
Execution of Knuth's Algorithm



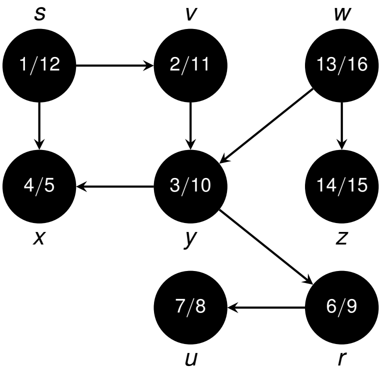
Execution of Knuth's Algorithm



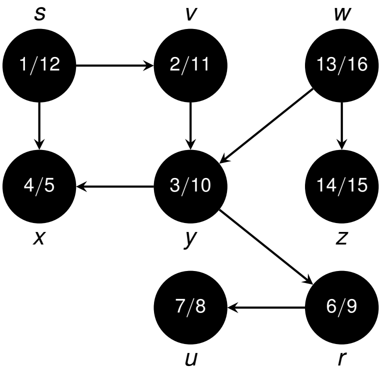
Execution of Knuth’s Algorithm



Execution of Knuth's Algorithm



Execution of Knuth's Algorithm



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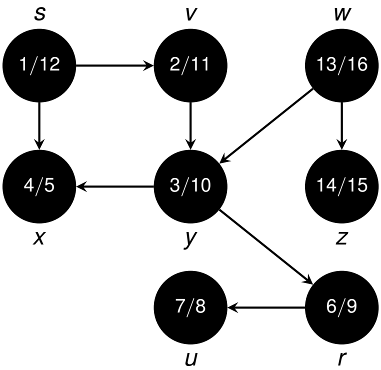

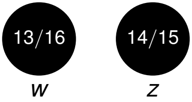

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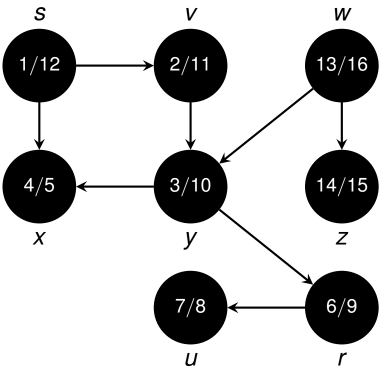

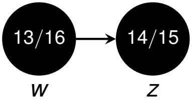

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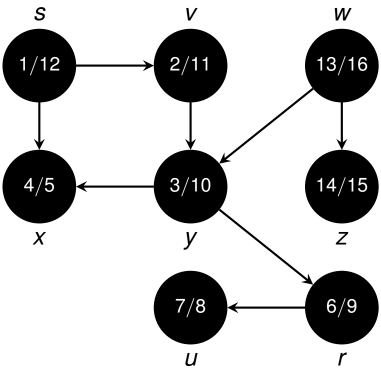
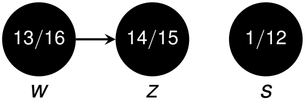
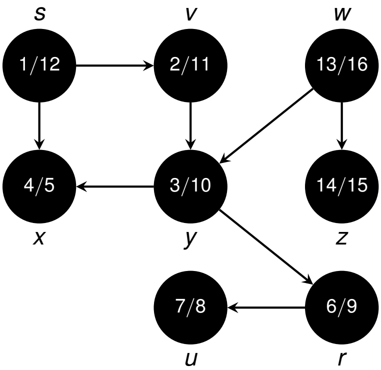
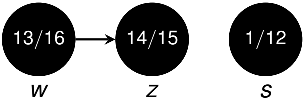




Execution of Knuth's Algorithm





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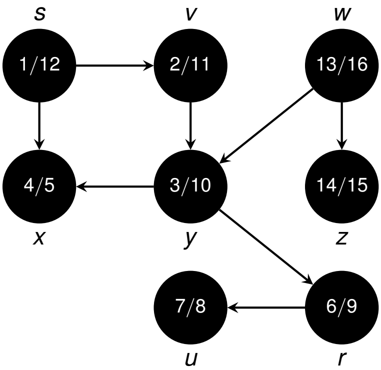

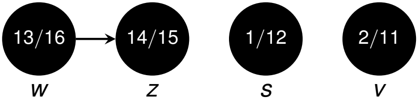

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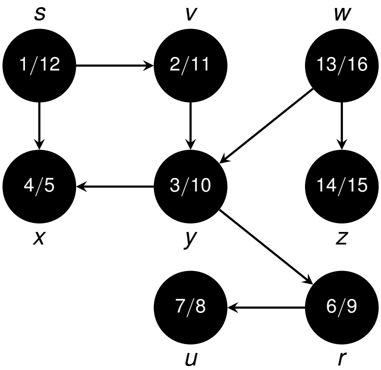

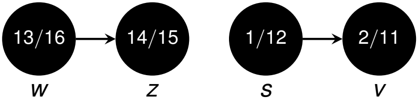

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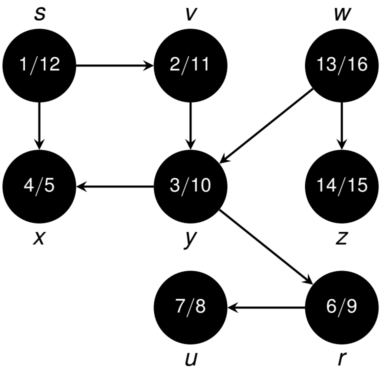

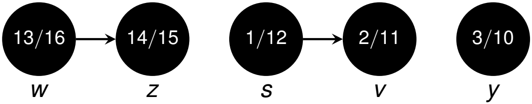

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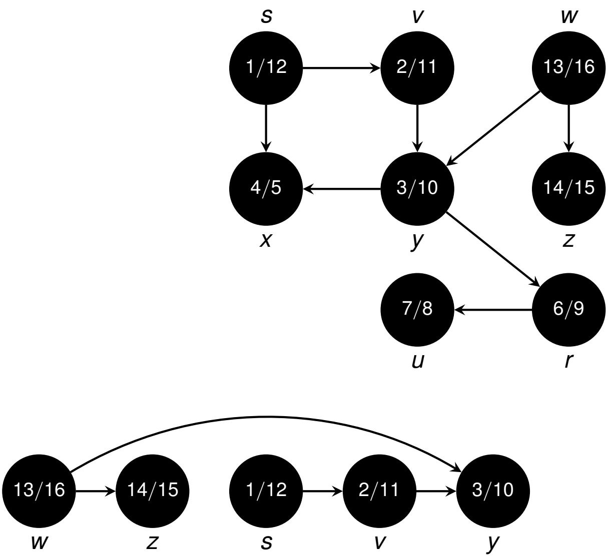

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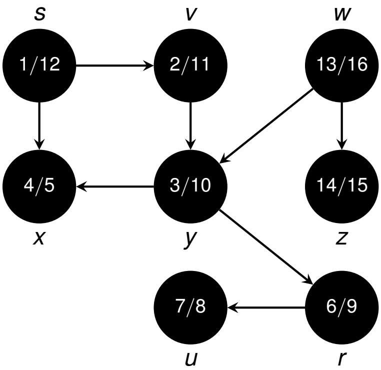

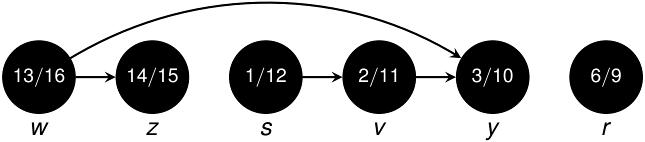

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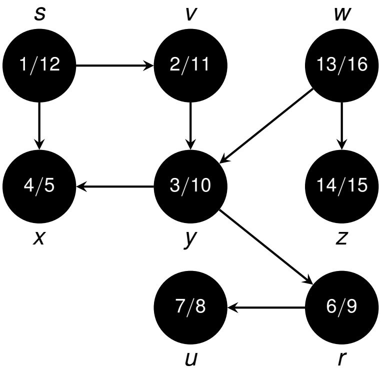

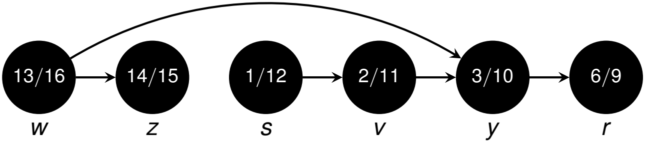

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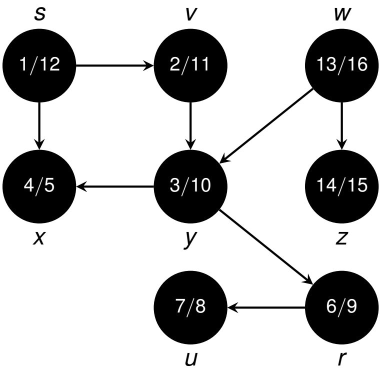

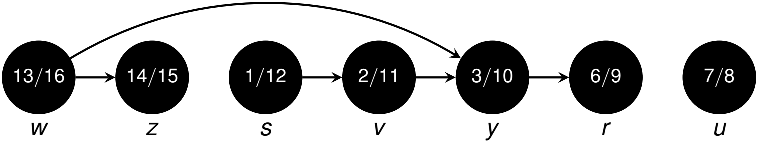

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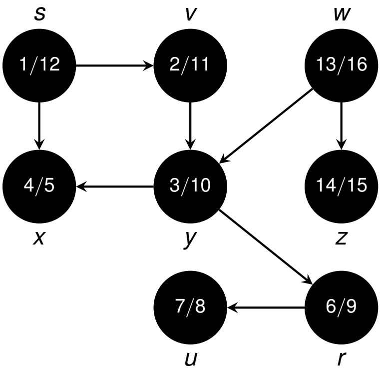

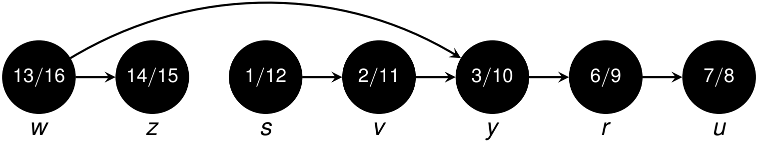

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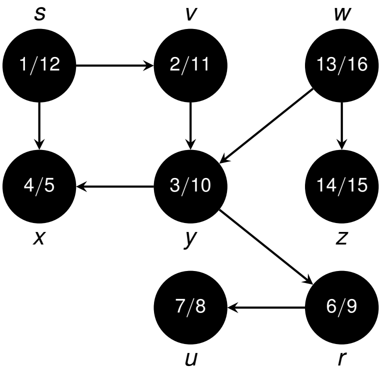

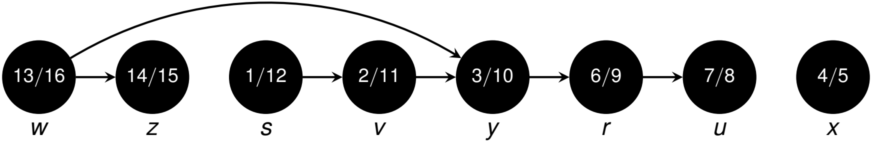

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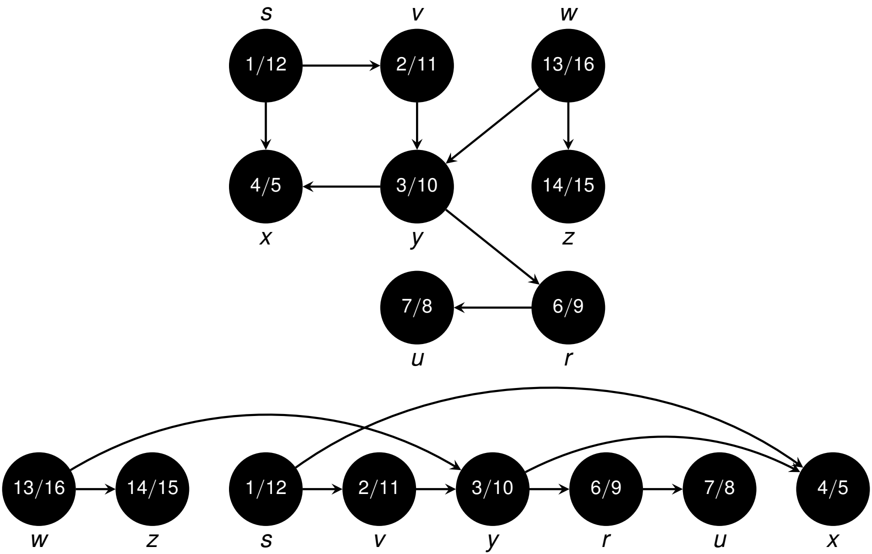
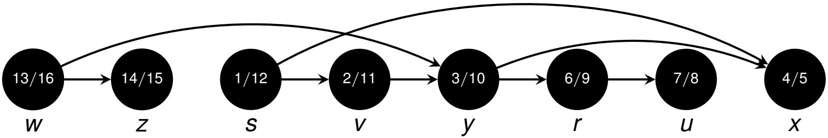


Correctness of Topological Sort using DFS



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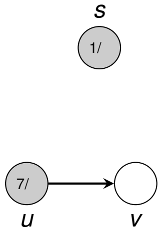

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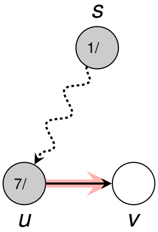

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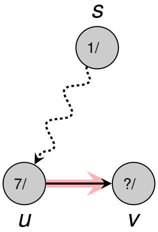

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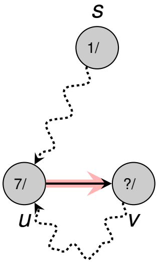

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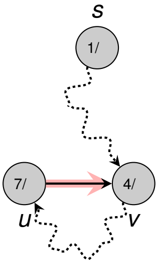

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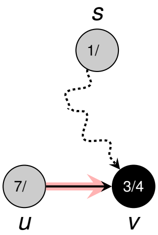

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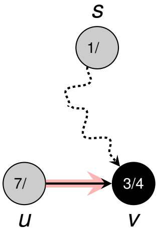

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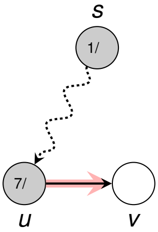

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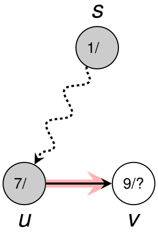

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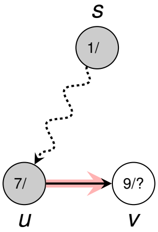

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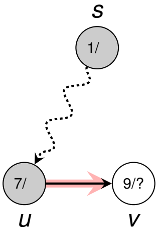

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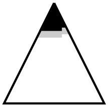
Summary of Graph Searching

Breadth-First-Search

- vertices are processed by a queue
- computes distances and shortest paths
  ⇝ similar idea used later in Prim's and Dijkstra's algorithm
- Runtime $\mathcal{O}(V + E)$



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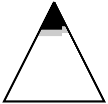
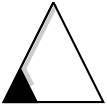


Depth-First-Search

- vertices are processed by recursive calls ($\approx$ stack)
- discovery and finishing times
- application: Topogical Sorting of DAGs
- Runtime $\mathcal{O}(V + E)$



Outline

Breadth-First Search

Depth-First Search

Topological Sort

Minimum Spanning Tree Problem

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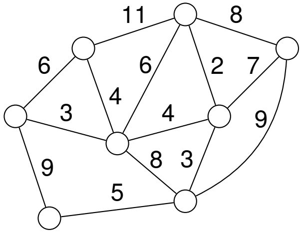

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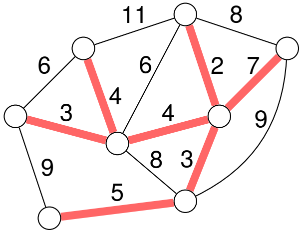

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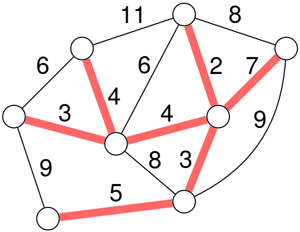

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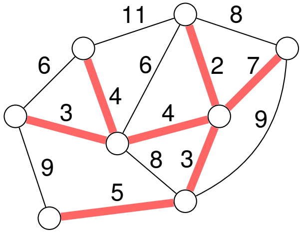

Applications

- Street Networks, Wiring Electronic Components, Laying Pipes
- Weights may represent distances, costs, travel times, capacities, resistance etc.

Generic Algorithm

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  A = empty set of edges
  while A does not span all vertices yet:
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How to find a safe edge?

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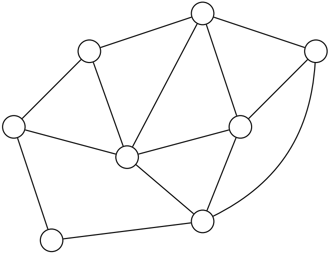

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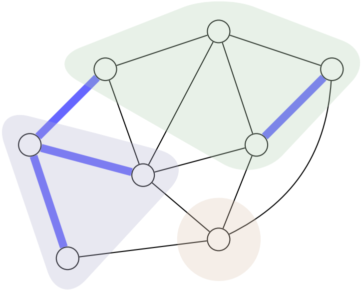

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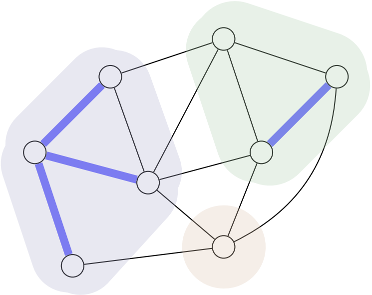

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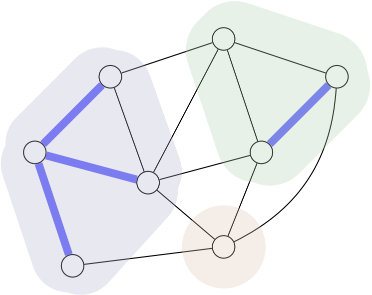

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Theorem

Let $A \subseteq E$ be a subset of a MST of G. Then for any cut that respects A, the lightest edge of G that goes across the cut is safe.