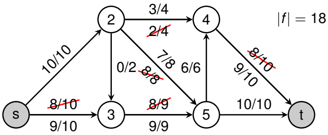
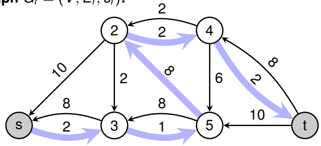
Graph $G = (V, E, c)$**:**



Residual Graph $G_f = (V, E_f, c_f)$**:**



6.6: Maximum flow

Frank Stajano

Thomas Sauerwald

Lent 2015

UNIVERSITY OF CAMBRIDGE

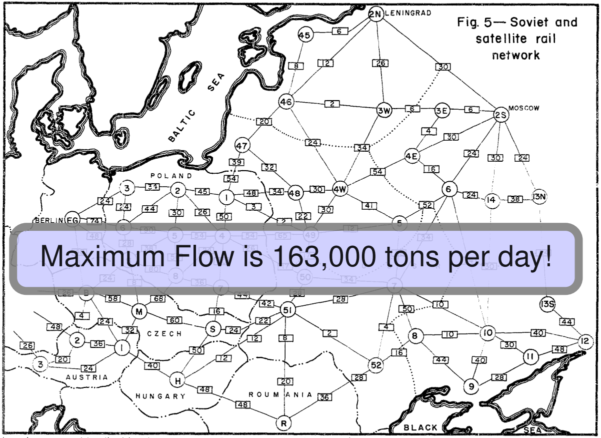
Outline

Recap

Max-Flow Min-Cut Theorem

Analysis of Ford-Fulkerson

History of the Maximum Flow Problem [Harris, Ross (1955)]



Legend: —··— International boundary ·········· Regional boundaries of the USSR (they are included as a matter of general information)

(7) Operating divisions. Those located in Russia are believed to be accurately located. Some Russian divisions (2, 3, 4 and 13) are located in two regions and are so indicated. Divisions shown in the satellites are indicated according to the authors' best judgment, since intelligence reports are unavailable. Train capacities in Russia are for 1000-net-ton trains or their equivalent. Train capacities in Poland are for 666 net tons (or the equivalent). Train capacities in all other satellites are for 400 net tons (or the equivalent) except in East Germany. In East Germany, train capacities are those of entering lines. The numbers shown in boxes are total interdivisional capacities.

Multiple Sources and Multiple Sinks (Figure 26.1)

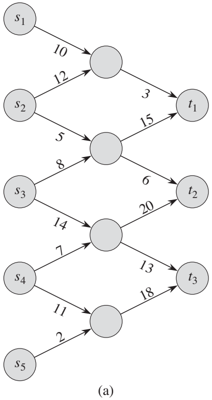
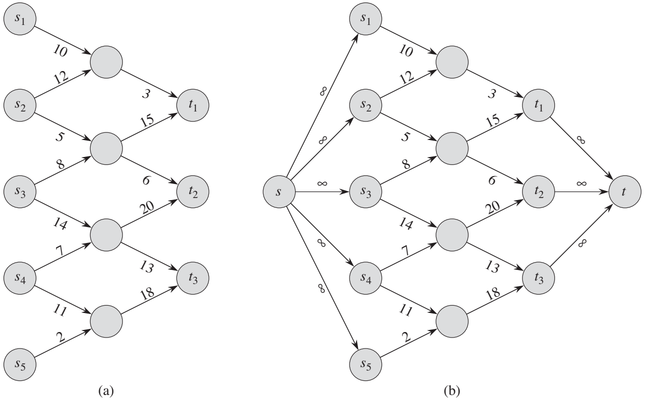


Figure 26.3 Converting a multiple-source, multiple-sink maximum-flow problem into a problem with a single source and a single sink. **(a)** A flow network with five sources $S = \{s_1, s_2, s_3, s_4, s_5\}$ and three sinks $T = \{t_1, t_2, t_3\}$. **(b)** An equivalent single-source, single-sink flow network. We add a supersource s and an edge with infinite capacity from s to each of the multiple sources. We also add a supersink t and an edge with infinite capacity from each of the multiple sinks to t.

Multiple Sources and Multiple Sinks (Figure 26.1)



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Residual Graph

Original Edge

Edge $e = (u, v) \in E$

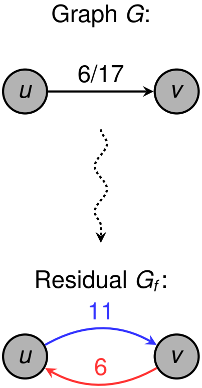
- flow $f(u, v)$ and capacity $c(u, v)$

Residual Capacity

$$c_f(u, v) = \begin{cases} c(u, v) - f(u, v) & \text{if } (u, v) \in E, \\ f(v, u) & \text{if } (v, u) \in E, \\ 0 & \text{otherwise.} \end{cases}$$

Residual Graph

- $G_f = (V, E_f, c_f)$, $E_f := \{(u, v) \colon c_f(u, v) > 0\}$



Residual Graph with anti-parallel edges

Original Edge

Edge $e = (u, v) \in E$ (& possibly $e' = (v, u) \in E$)

- flow $f(u, v)$ and capacity $c(u, v)$

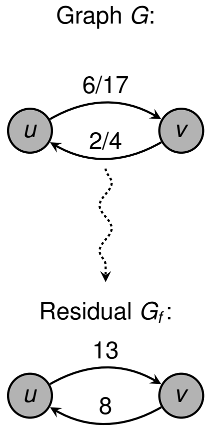
Residual Capacity

For every pair $(u, v) \in V \times V$,

$$c_f(u, v) = c(u, v) - f(u, v).$$

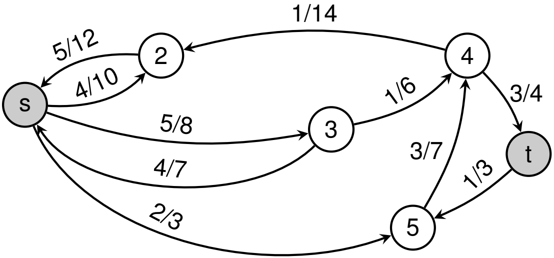
Residual Graph

- $G_f = (V, E_f, c_f)$, $E_f := \{(u, v) \colon c_f(u, v) > 0\}$

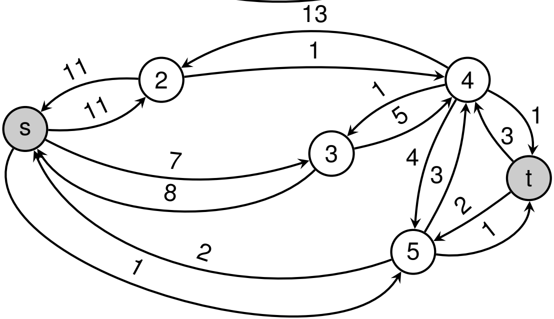


Example of a Residual Graph (Handout)

Flow network G



Residual Graph G_f



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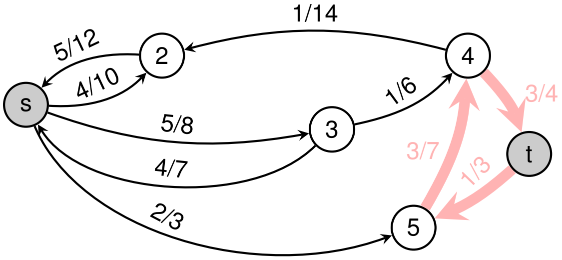

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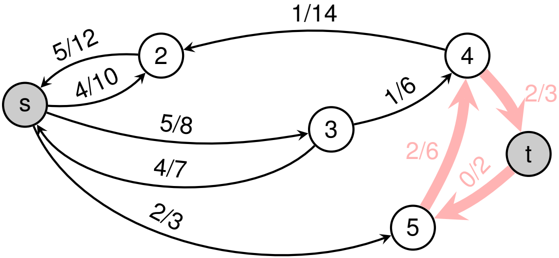

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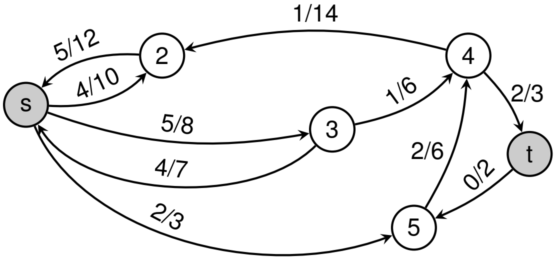

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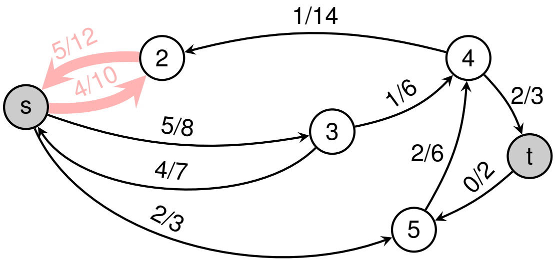

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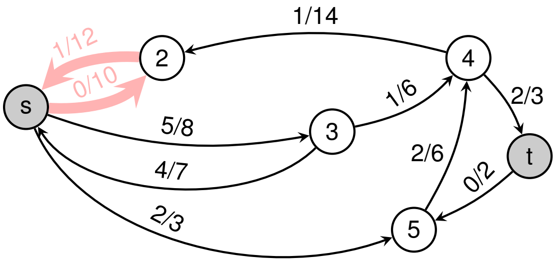
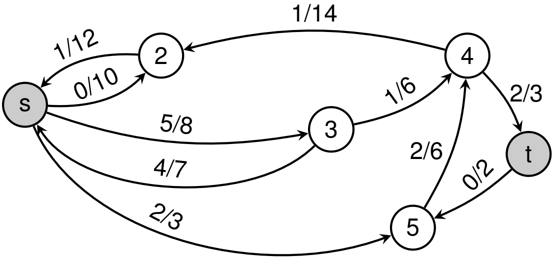


Example of a Residual Graph (Handout)

Flow network *G*



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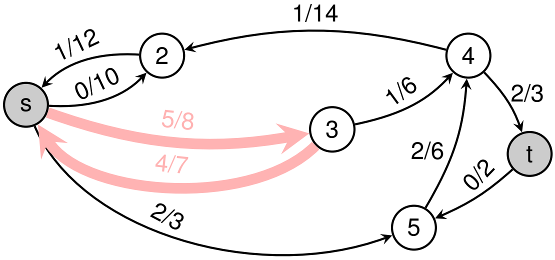

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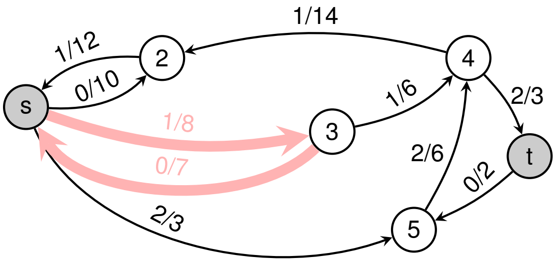

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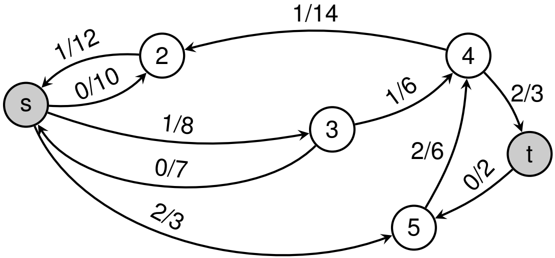

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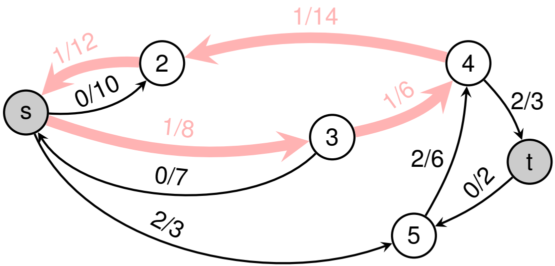

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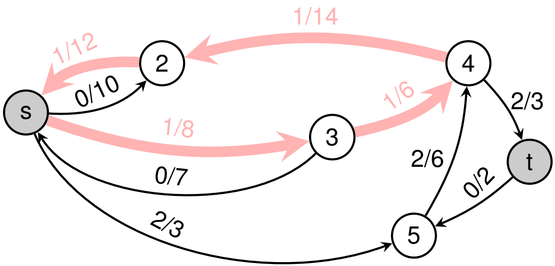

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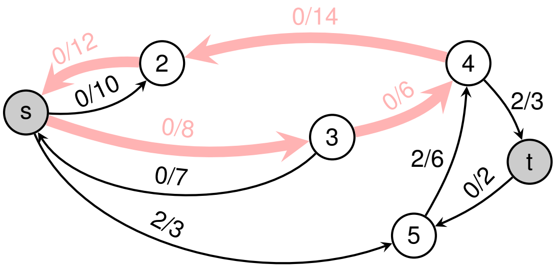

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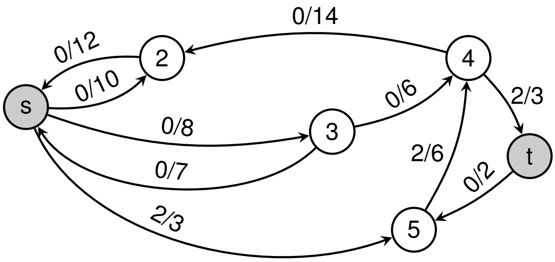

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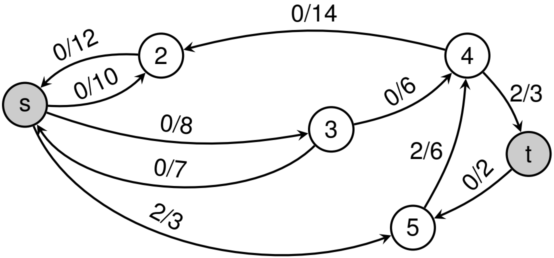
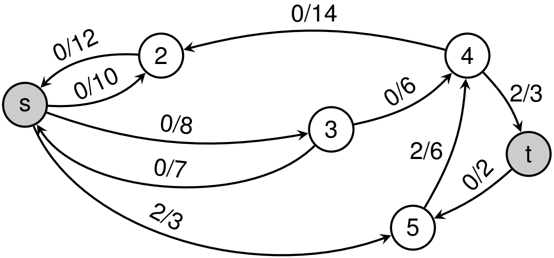


By successively eliminating cycles we can simplify and reduce the "transportation" cost of a flow.

Example of a Residual Graph (Handout)

Flow network G



Comment on anti-parallel edges:

- You should know about this possibility
- In the following proofs we will disallow anti-parallel edges (for convenience)

The Ford-Fulkerson Method ("Enhanced Greedy")

```
def fordFulkerson(G)
    initialize flow to 0 on all edges
    while an augmenting path in G_f can be found:
        push as much extra flow as possible through it
```

Augmenting path: Path from source to sink in G_f

Questions:

- How to find an augmenting path? ✓
- Does this method terminate?
- If it terminates, how good is the solution?

Outline

Recap

Max-Flow Min-Cut Theorem

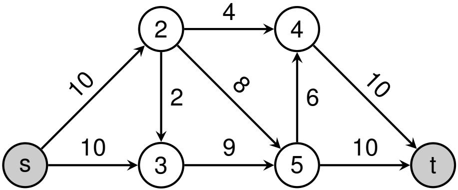
Analysis of Ford-Fulkerson

From Flows to Cuts

Cut

- A cut (S, T) is a partition of V into S and $T = V \setminus S$ such that $s \in S$ and $t \in T$.

Graph $G = (V, E, c)$**:**



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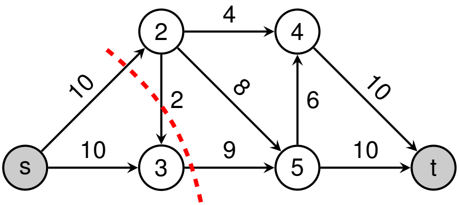

From Flows to Cuts

Cut

- A cut (S, T) is a partition of V into S and $T = V \setminus S$ such that $s \in S$ and $t \in T$.
- The capacity of a cut (S, T) is the sum of capacities of the edges from S to T:

$$\mathrm{c}(S,T) = \sum_{u \in S, v \in T} c(u,v) = \sum_{(u,v) \in E(S,T)} c(u,v)$$

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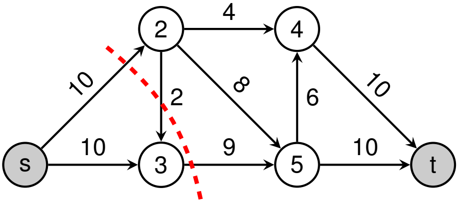

$\mathrm{c}(\{s, 3\}, \{2, 4, 5, t\}) =$

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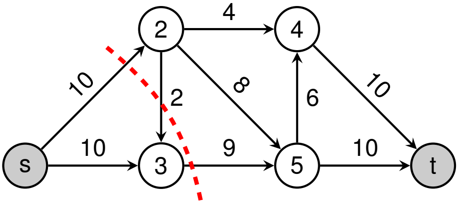

$$c(\{s,3\},\{2,4,5,t\}) = 10 + 9 = 19$$

From Flows to Cuts

Cut

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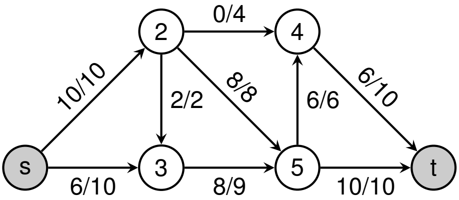
$$c(S, T) = \sum_{u \in S, v \in T} c(u, v) = \sum_{(u,v) \in E(S,T)} c(u, v)$$

- A mininum cut of a network is a cut whose capacity is minimum over all cuts of the network.

Graph $G = (V, E, c)$**:** $|f| = 16$



From Flows to Cuts

Flow Value Lemma (Lemma 26.4)

Let f be a flow with source s and sink t, and let (S, T) be any cut of G. Then the value of the flow is equal to the net flow across the cut, i.e.,

$$|f| = \sum_{(u,v)\in E(S,T)} f(u,v) - \sum_{(v,u)\in E(T,S)} f(v,u).$$

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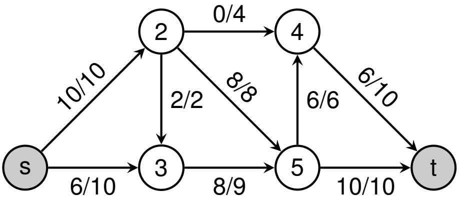

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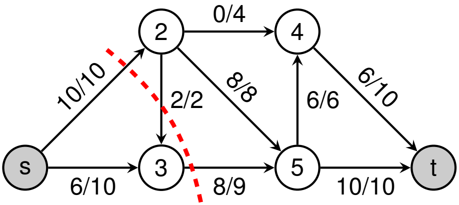

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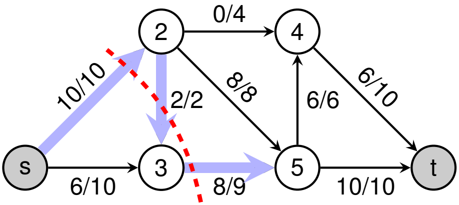

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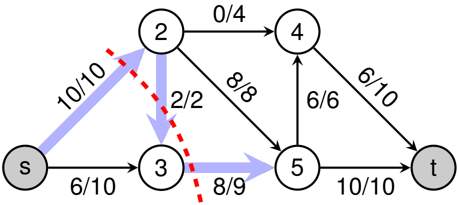

$10 - 2 + 8 = 16$

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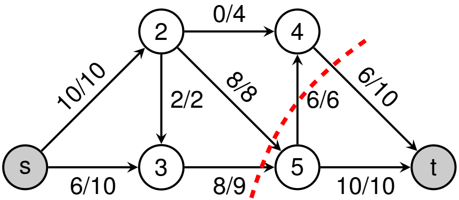

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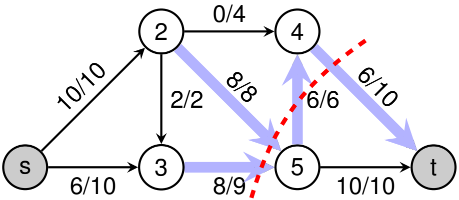

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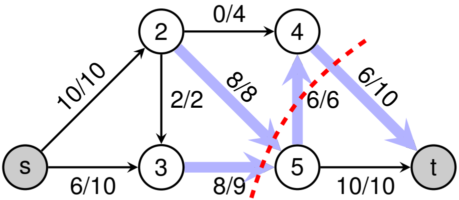

$8 + 8 - 6 + 6 = 16$

From Flows to Cuts

$$|f| = \sum_{(u,v)\in E(S,T)} f(u,v) - \sum_{(v,u)\in E(T,S)} f(v,u).$$

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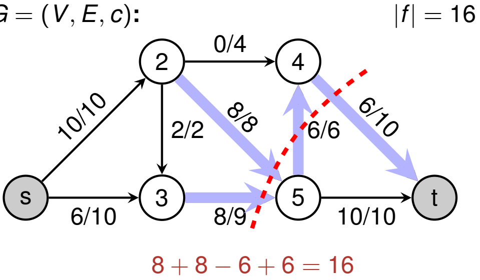

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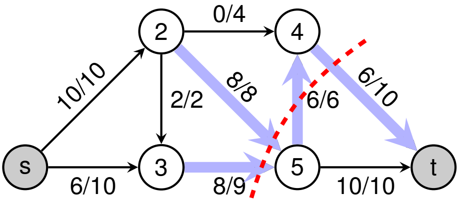

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From Flows to Cuts

$$|f| = \sum_{(u,v)\in E(S,T)} f(u,v) - \sum_{(v,u)\in E(T,S)} f(v,u).$$

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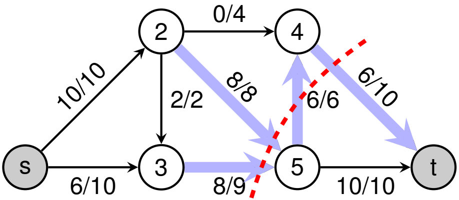
Graph $G = (V, E, c)$**:** $|f| = 16$



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$$|f| = \sum_{(u,v)\in E(S,T)} f(u,v) - \sum_{(v,u)\in E(T,S)} f(v,u).$$

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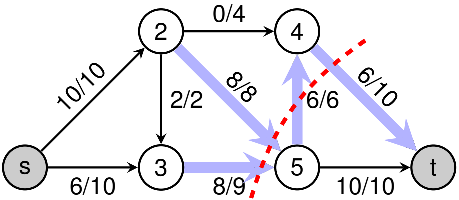

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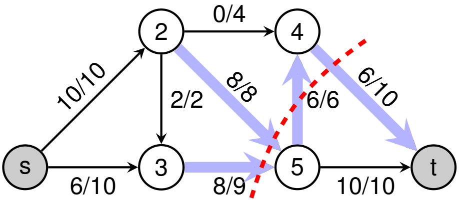
$$|f| = \sum_{(u,v)\in E(S,T)} f(u,v) - \sum_{(v,u)\in E(T,S)} f(v,u).$$

$$|f| = \sum_{w\in V} f(s,w) = \sum_{u\in S}\left(\sum_{(u,w)\in E} f(u,w) - \sum_{(w,u)\in E} f(w,u)\right)$$

$$= \sum_{(u,v)\in E(S,T)} f(u,v) - \sum_{(v,u)\in E(T,S)} f(v,u) \qquad \square$$

Graph $G = (V, E, c)$: $|f| = 16$



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Weak Duality betwen Flows and Cuts

Weak Duality (Corollary 26.5)

Let f be any flow and (S, T) be any cut.

Weak Duality betwen Flows and Cuts

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$$|f| \leq c(S, T).$$

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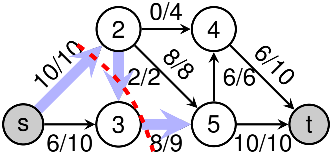

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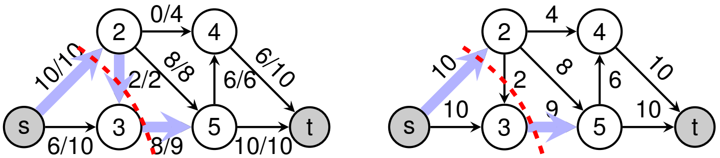

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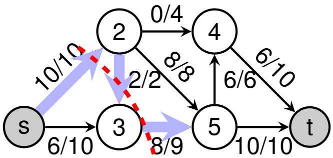

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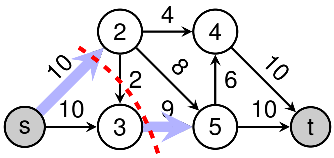

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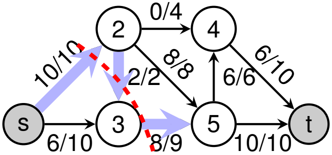

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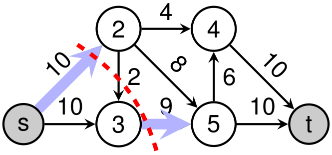

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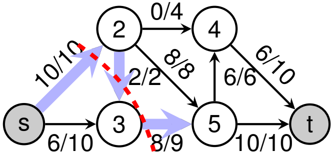

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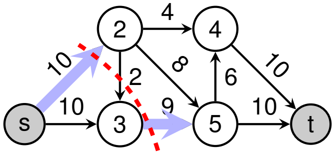

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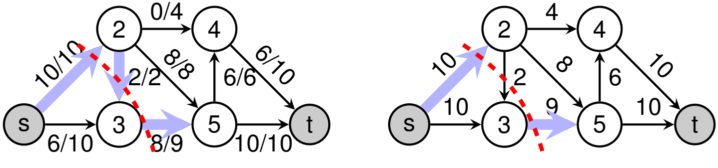

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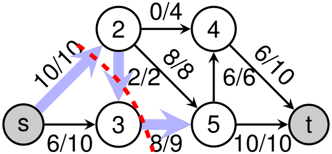

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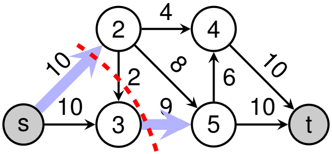

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Max-Flow Min-Cut Theorem

Theorem

The value of the max-flow is equal to the capacity of the min-cut, that is

$$\max_f |f| = \min_{S,T \subseteq V} c(S,T).$$

Key Lemma

Key Lemma (Theorem 26.6)

The following three conditions are all equivalent for any flow f:

1. f is a maximum flow
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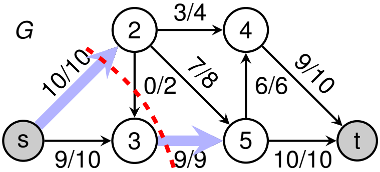

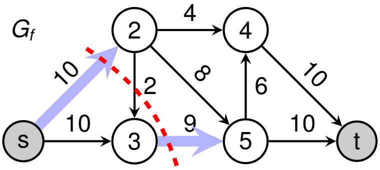

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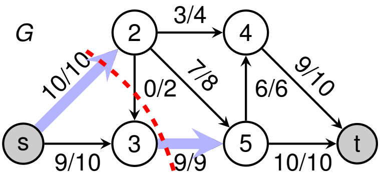

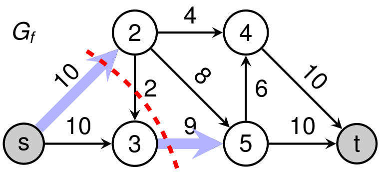

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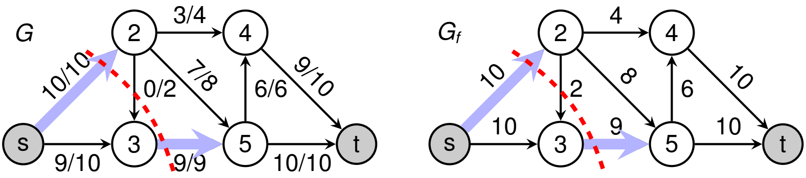

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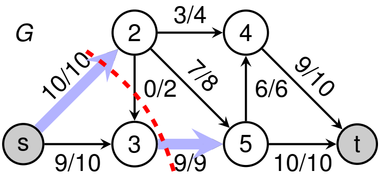
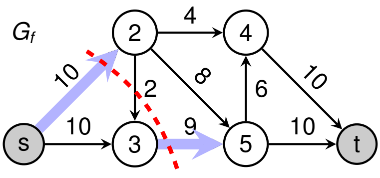
- Hence f is a maximum flow.





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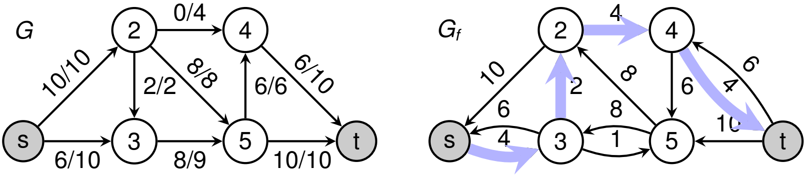

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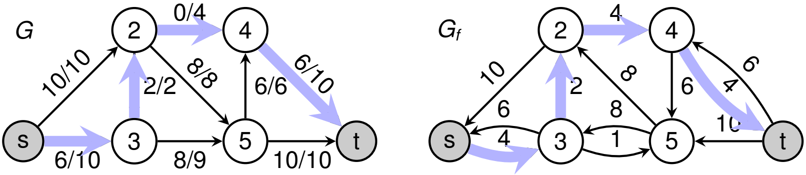

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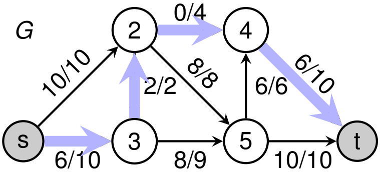

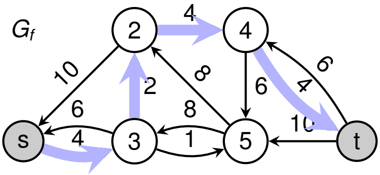

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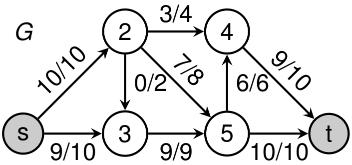

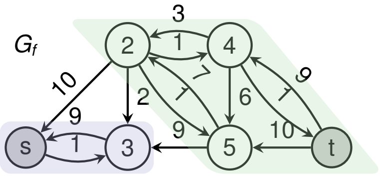

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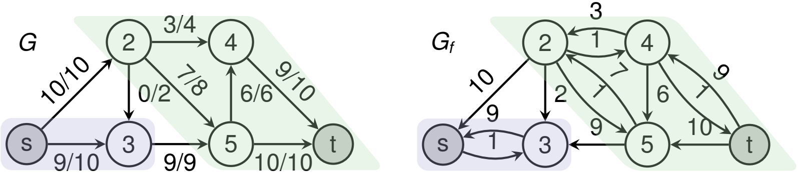

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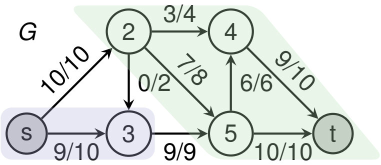

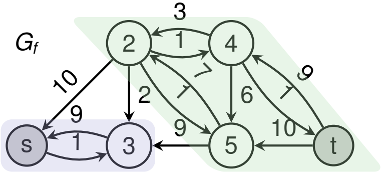

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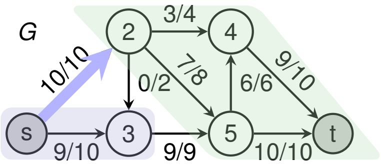

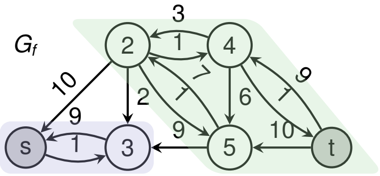

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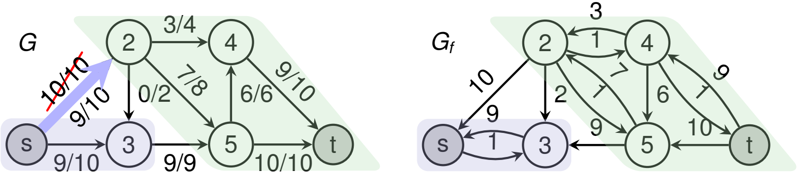

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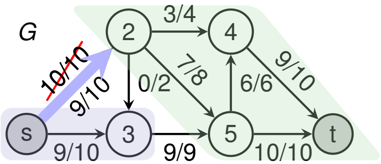

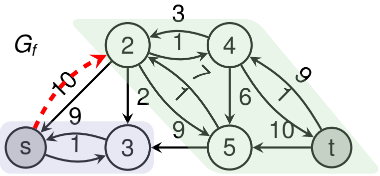

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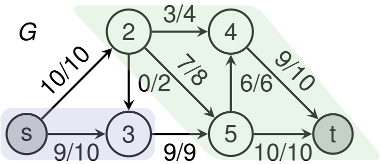

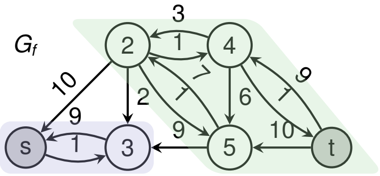

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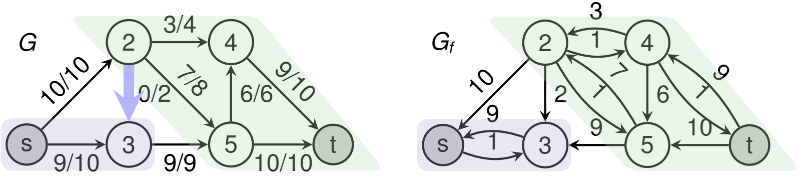

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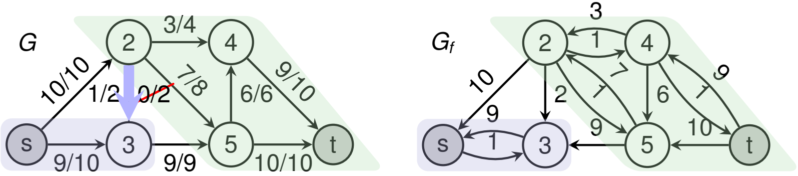

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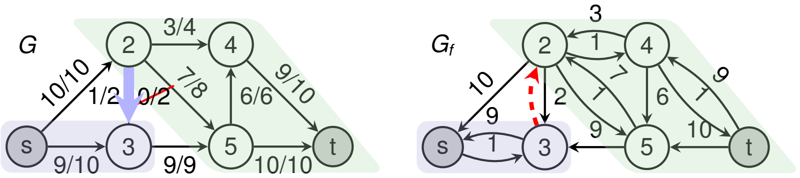

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- $(v, u) \in E(T, S) \Rightarrow f(v, u) = 0$.

$$|f| = \sum_{(u,v)\in E(S,T)} f(u, v) - \sum_{(v,u)\in E(T,S)} f(v, u)$$

Key Lemma

Key Lemma (Theorem 26.6)

The following three conditions are all equivalent for any flow f:

1. f is a maximum flow
2. There is no augmenting path in G_f
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Proof 2 $\Rightarrow$ 3:

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Proof of the Max-Flow Min-Cut Theorem

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$\Rightarrow$

$$\max_f |f| = |f_{\max}| = c(S, T) \geq \min_{S,T \subseteq V} c(S, T)$$

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- Run Ford-Fulkerson until it terminates with a flow $f_{\max}$
- Define S as the set of nodes reachable from s in G_f, and $T = V \setminus S$

$\Rightarrow$ (S, T) is a minimal (s, t)-cut whose capacity is equal to $|f_{\max}|$

Outline

Recap

Max-Flow Min-Cut Theorem

Analysis of Ford-Fulkerson

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```
def FordFulkerson(G)
    initialize flow to 0 on all edges
    while an augmenting path in G_f can be found:
        push as much extra flow as possible through it
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Flow before iteration integral
& capacities in G_f are integral
$\Rightarrow$ Flow after iteration integral

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For integral capacities $c(u, v)$, Ford-Fulkerson terminates after $V \cdot C$ *iterations*, where $C := \max_{u,v} c(u, v)$ and returns the maximum flow.

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Theorem

For integral capacities $c(u, v)$, Ford-Fulkerson terminates after $V \cdot C$ *iterations*, where $C := \max_{u,v} c(u, v)$ and returns the maximum flow.

at the time of termination, no augmenting path
$\Rightarrow$ Ford-Fulkerson returns maxflow (Key Lemma)

Slow Convergence of Ford-Fulkerson (Figure 26.7)

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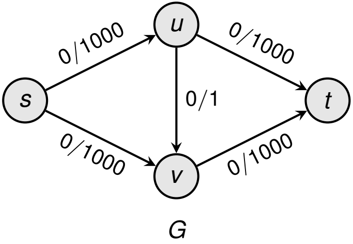

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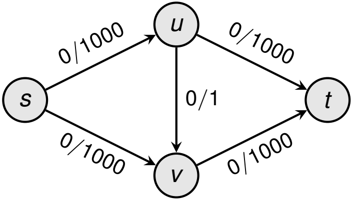

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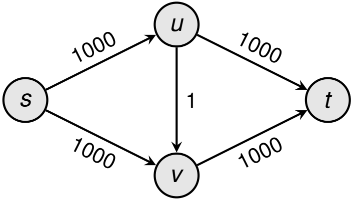

G_f

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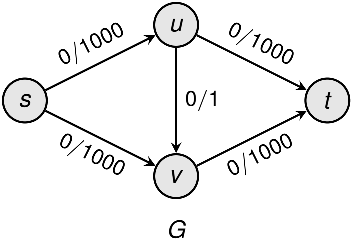

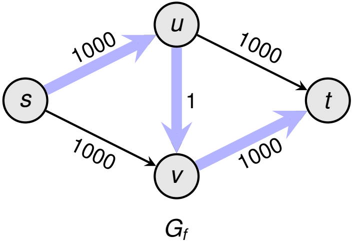

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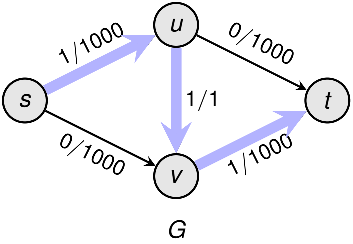

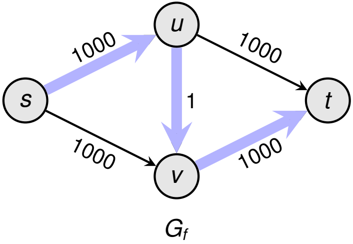

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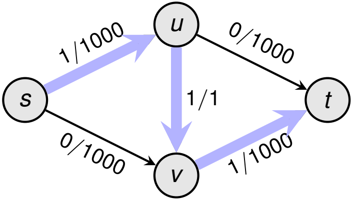

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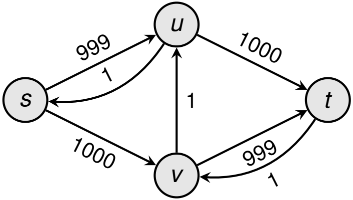

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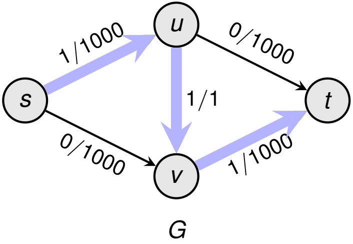

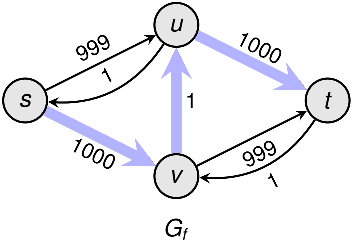

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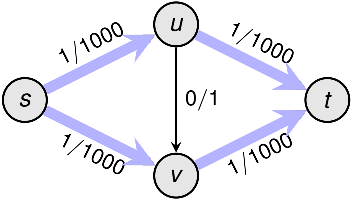

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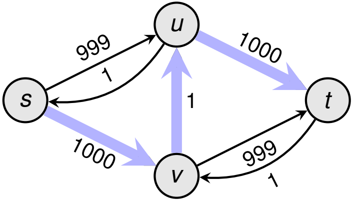

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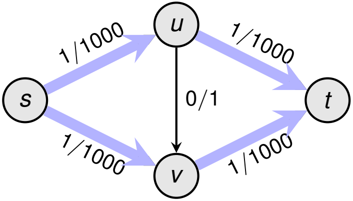

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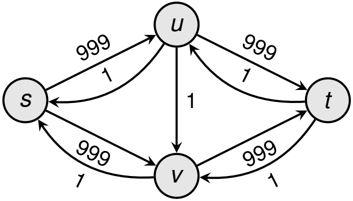

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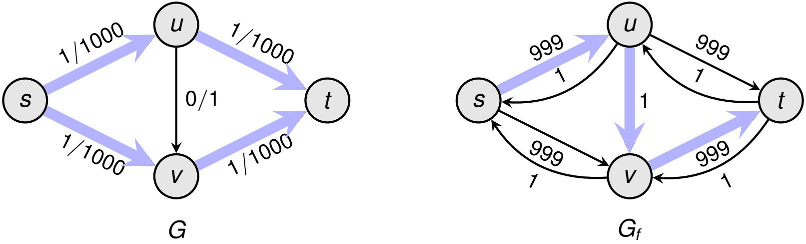

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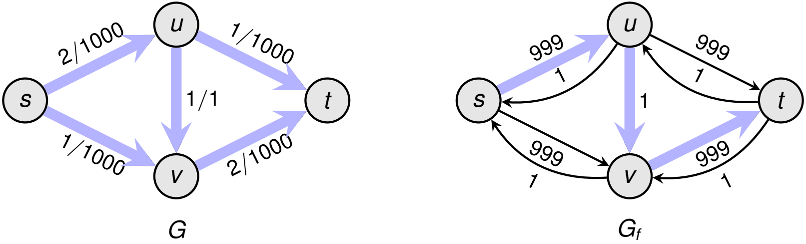

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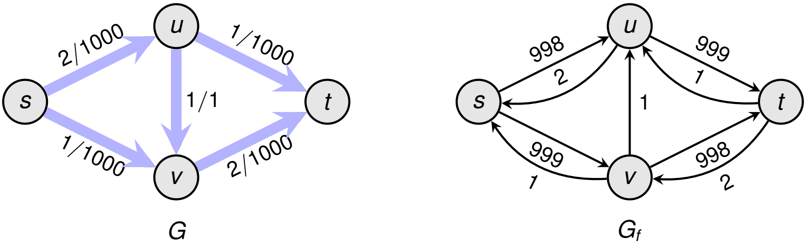

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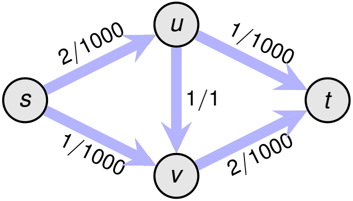

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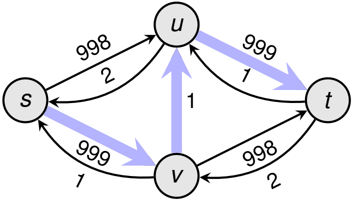

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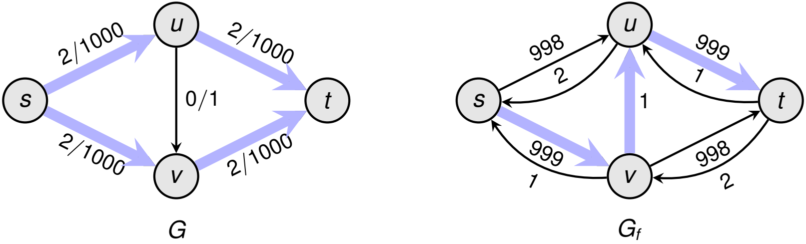

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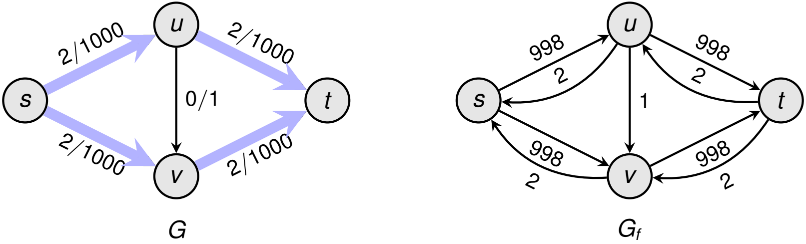

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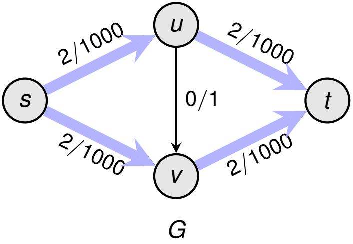

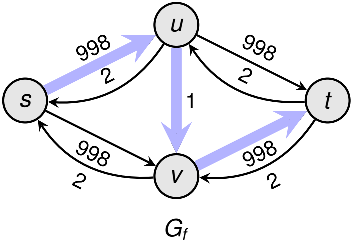

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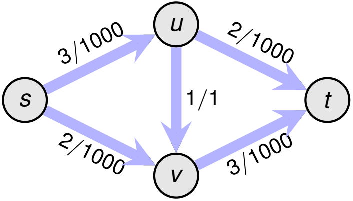

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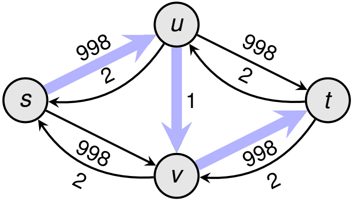

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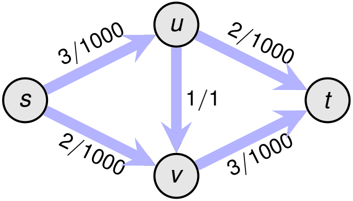

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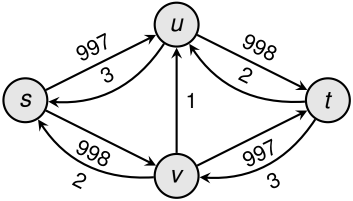

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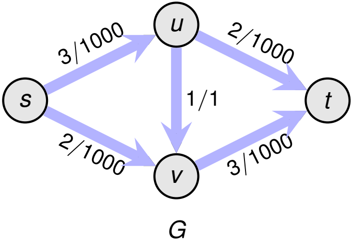

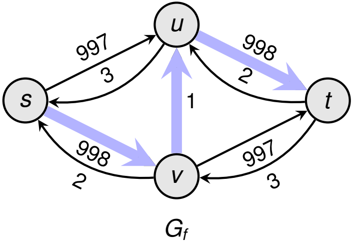

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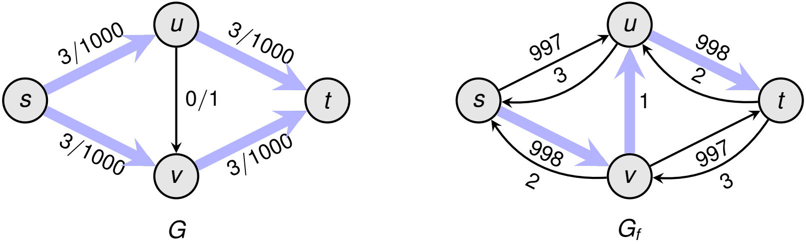

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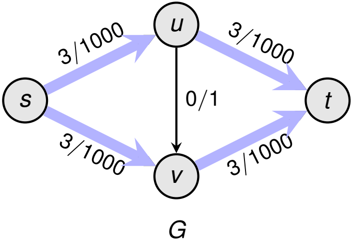

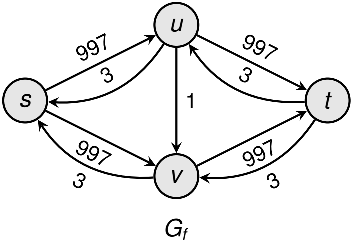

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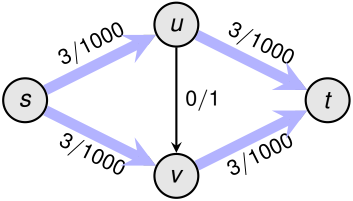

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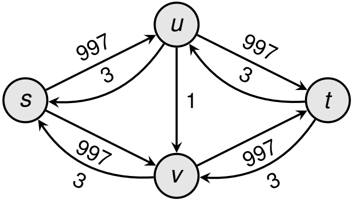

G_f

Number of iterations is at least $C := \max_{u,v} c(u, v)$!

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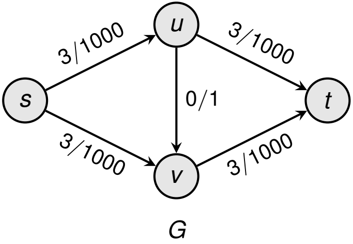
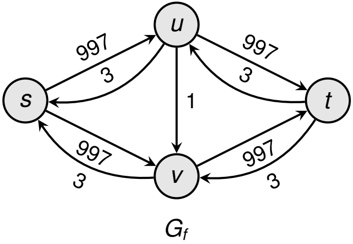
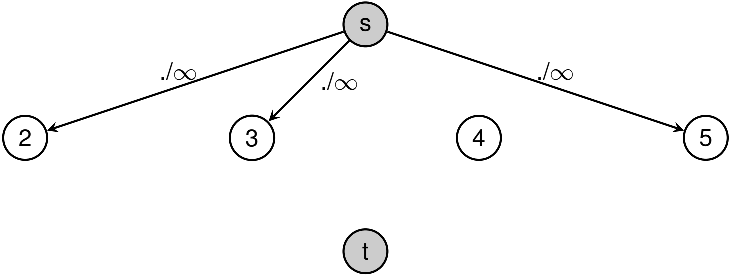




Number of iterations is at least $C := \max_{u,v} c(u,v)$!

For irrational capacities, Ford-Fulkerson may even fail to terminate!

Non-Termination of Ford-Fulkerson for Irrational Capacities



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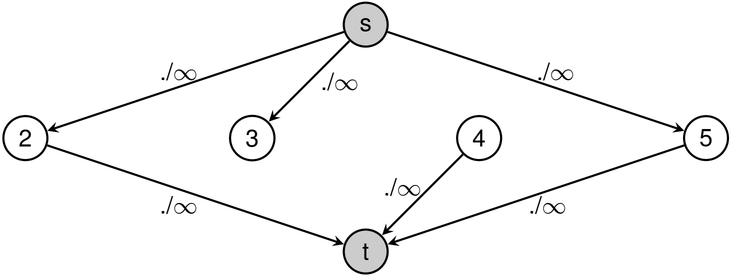

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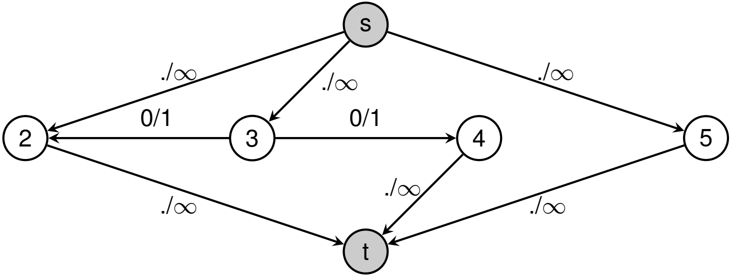

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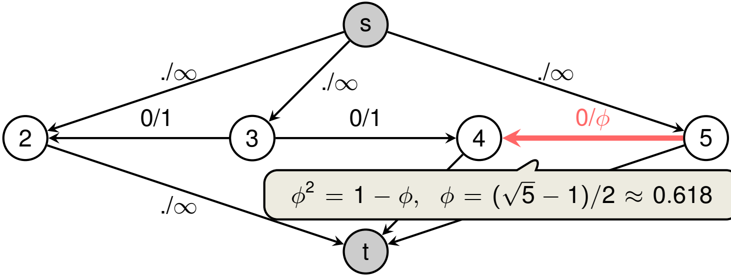
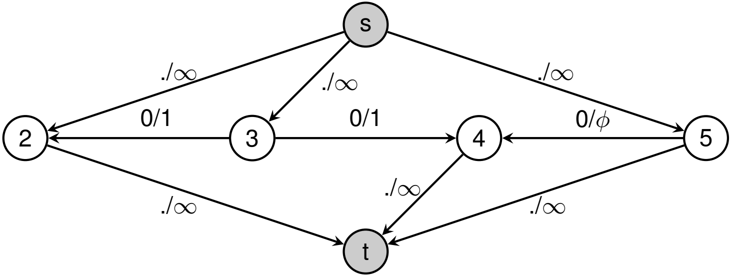


$\phi^2 = 1 - \phi, \quad \phi = (\sqrt{5} - 1)/2 \approx 0.618$

Non-Termination of Ford-Fulkerson for Irrational Capacities



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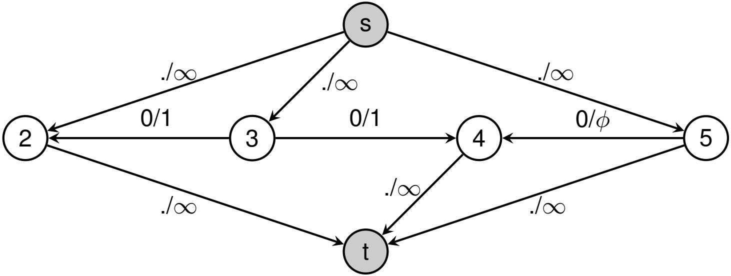

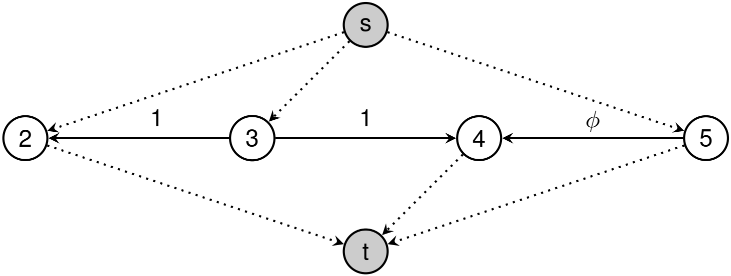

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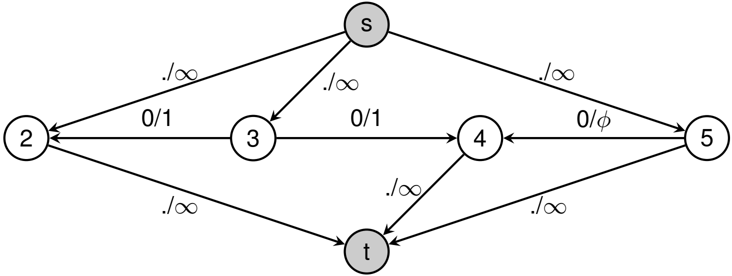
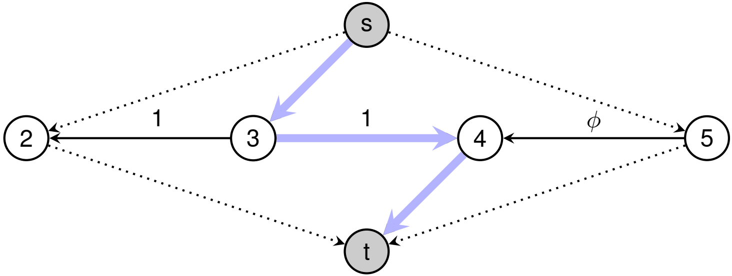


Iteration: 1, $|f| = 0$



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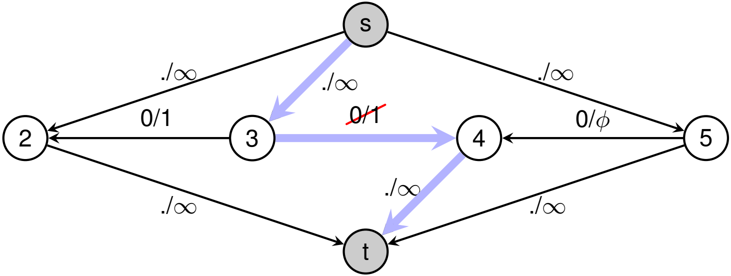

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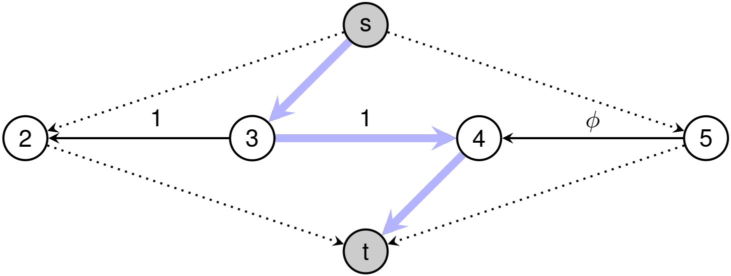

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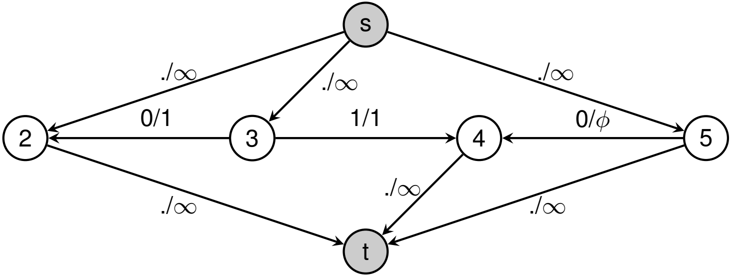
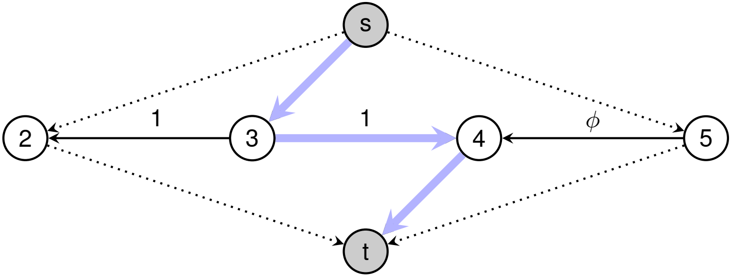


Iteration: 1, $|f| = 1$



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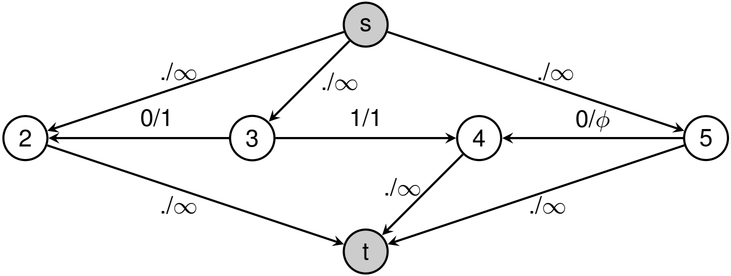

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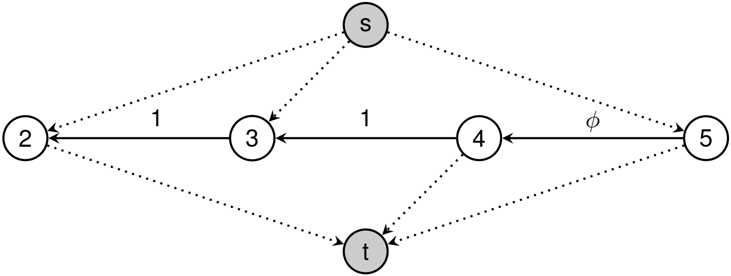

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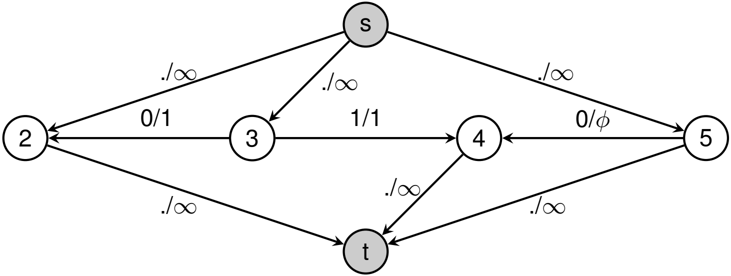
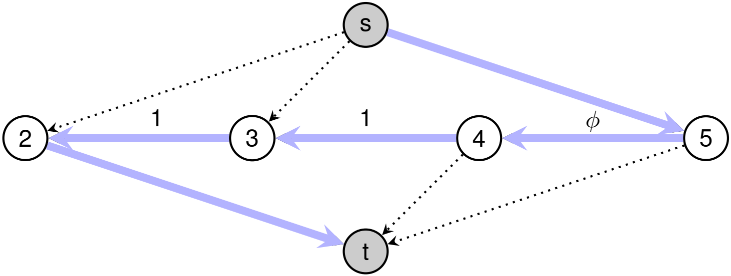


Iteration: 2, $|f| = 1$



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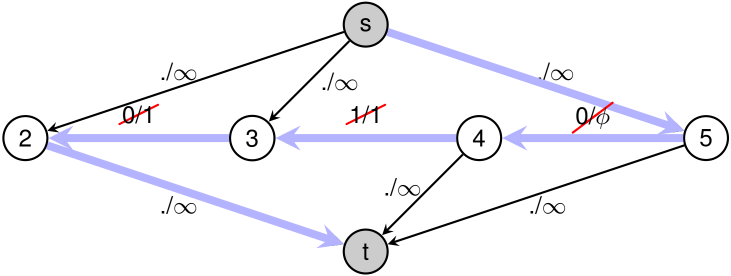

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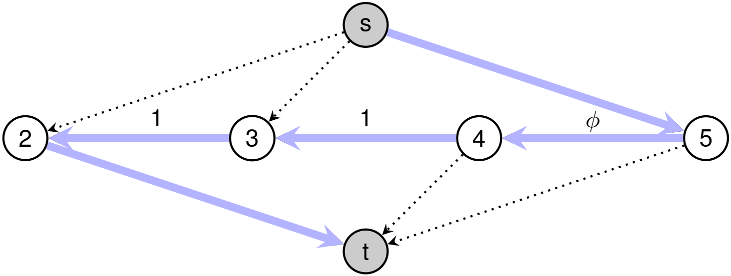

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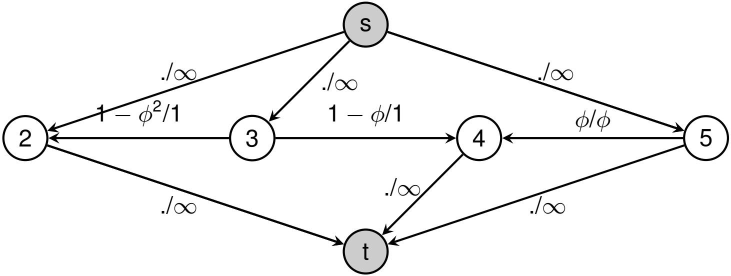
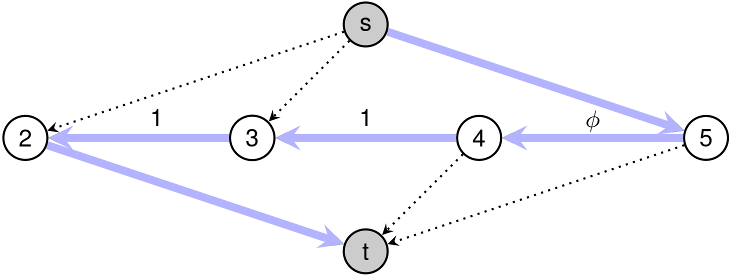


Iteration: 2, $|f| = 1 + \phi$



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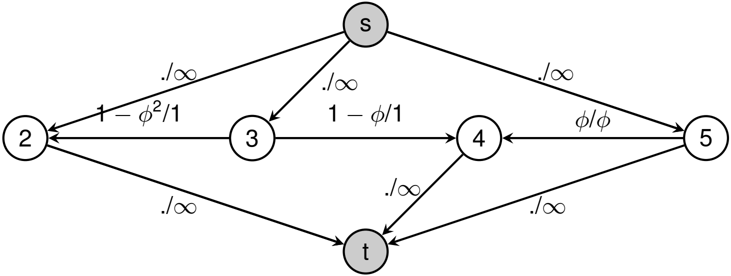

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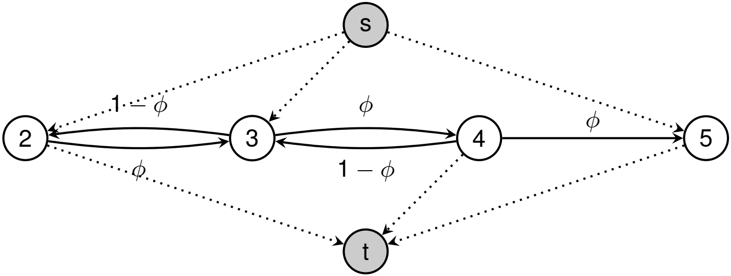

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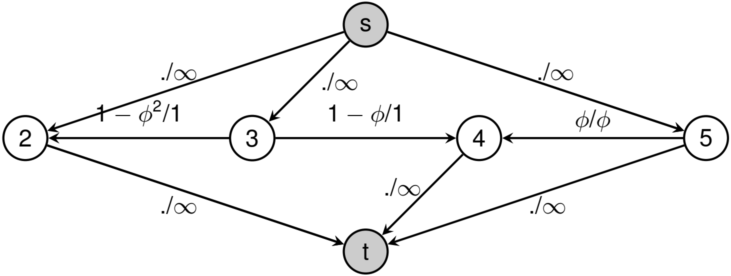
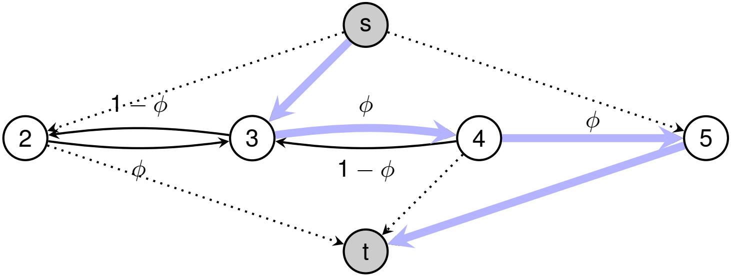


Iteration: 3, $|f| = 1 + \phi$



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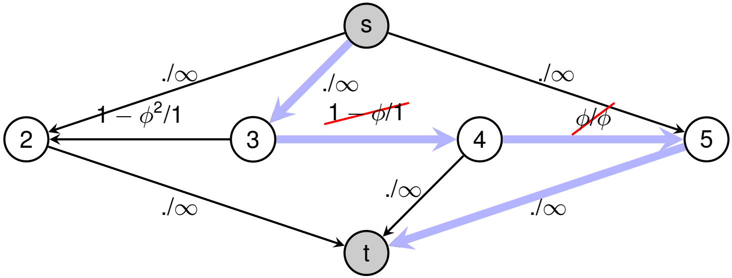

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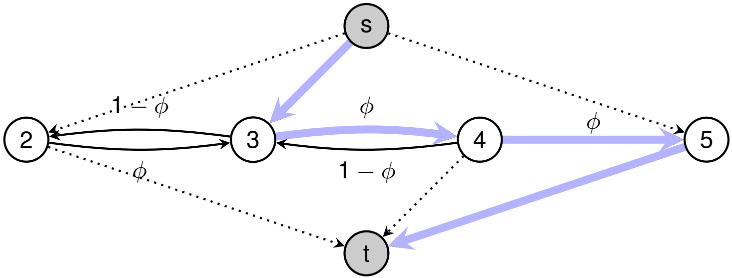

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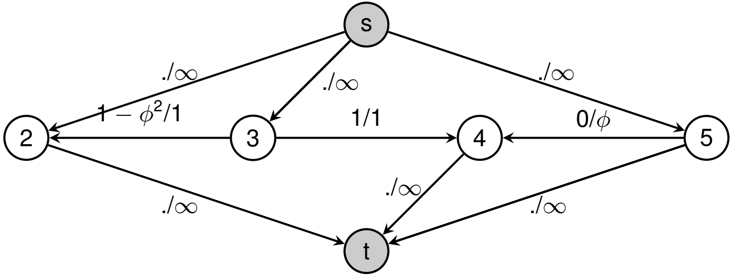
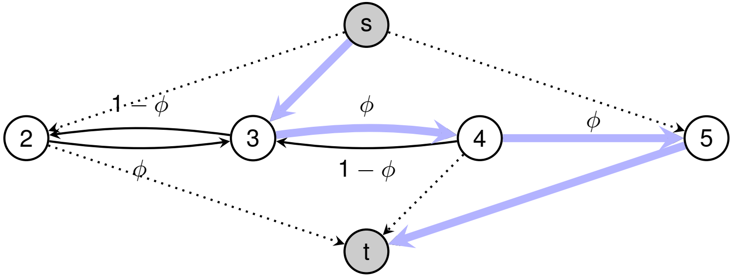


Iteration: 3, $|f| = 1 + 2 \cdot \phi$



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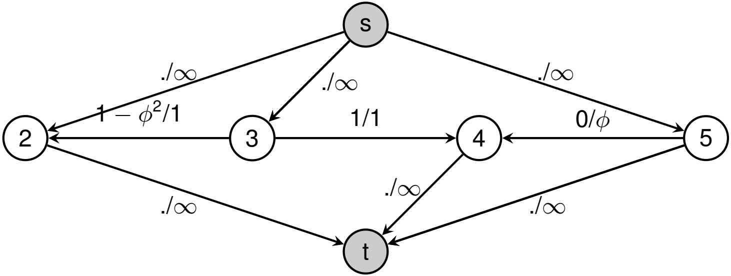

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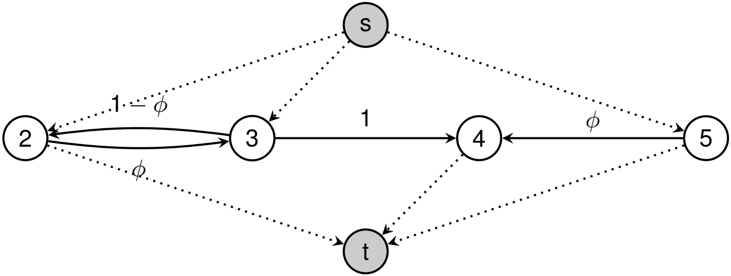

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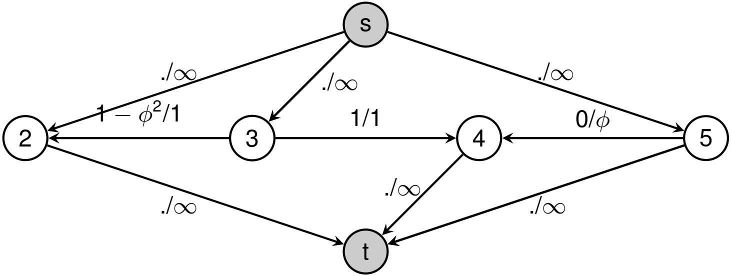

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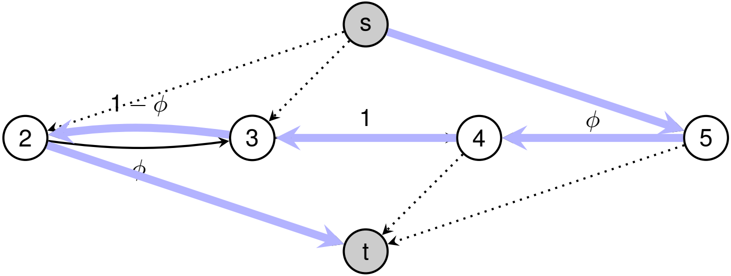

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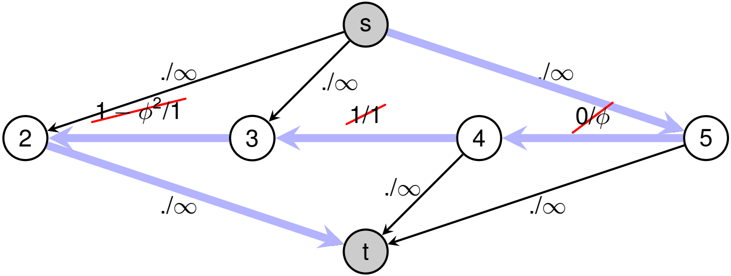

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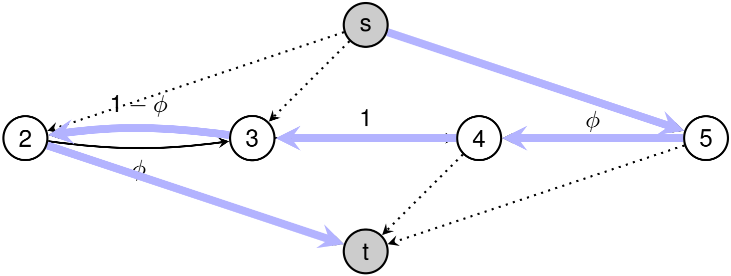

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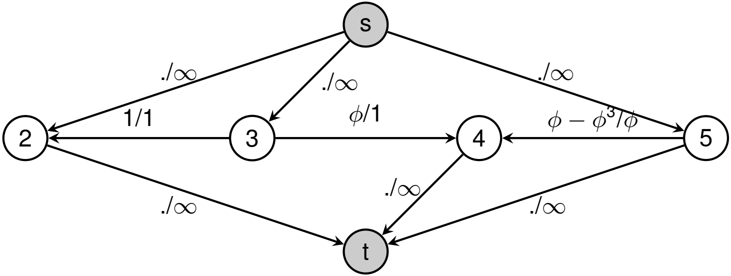
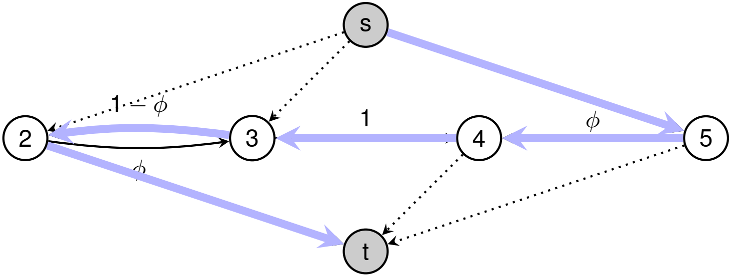


Iteration: 4, $|f| = 1 + 2 \cdot \phi + \phi^2$



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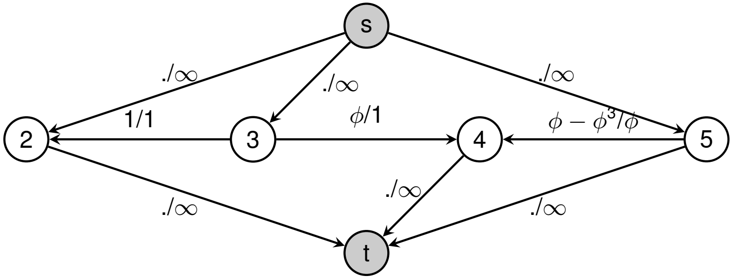

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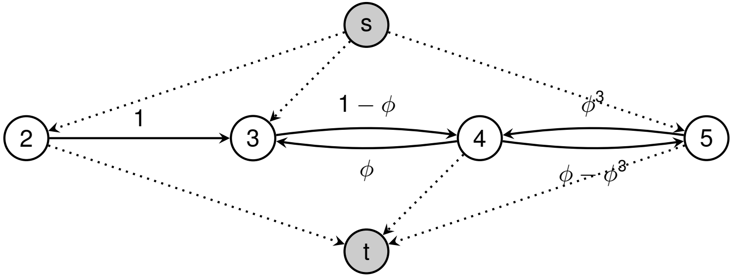

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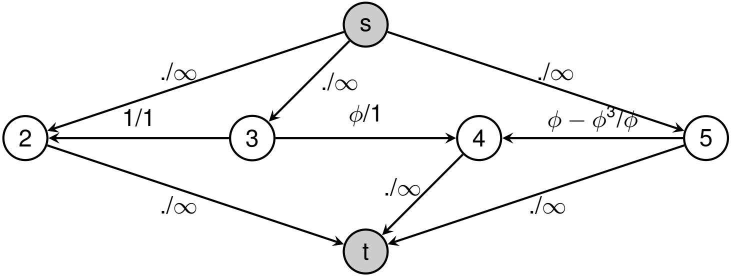
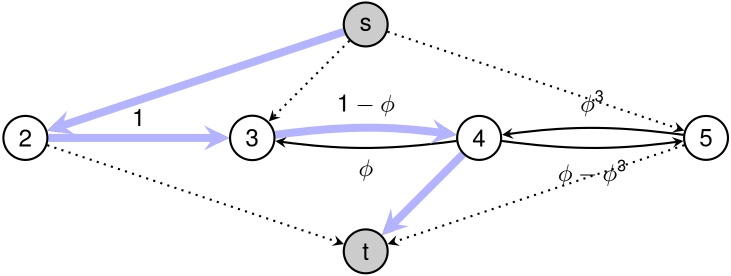


Iteration: 5, $|f| = 1 + 2 \cdot \phi + \phi^2$



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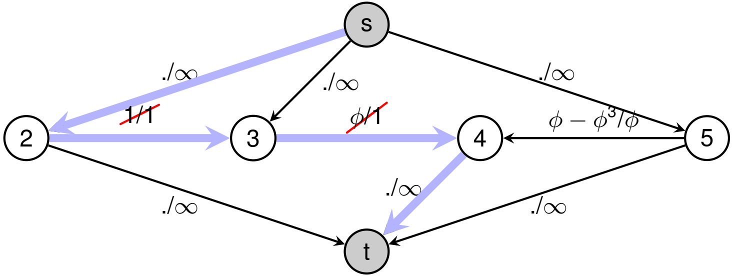
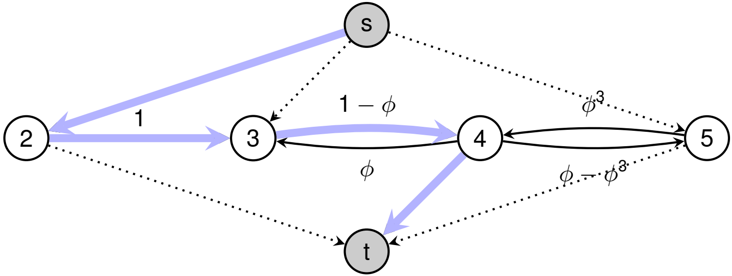


Iteration: 5, $|f| = 1 + 2 \cdot \phi + \phi^2$



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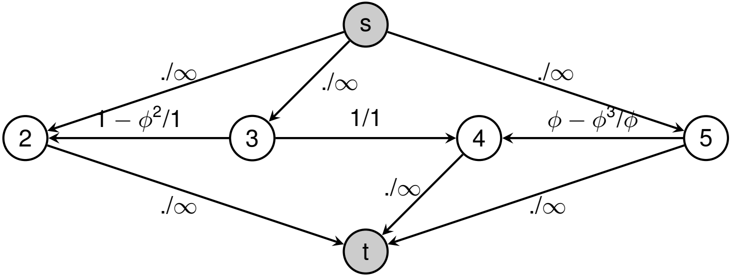
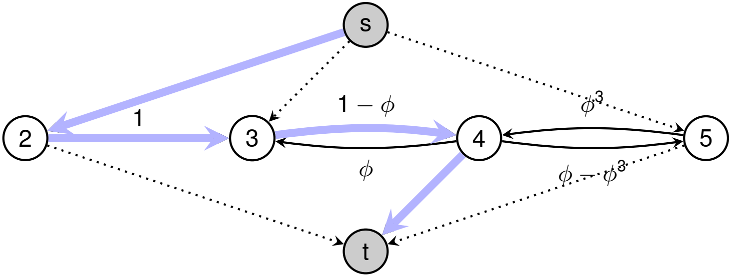


Iteration: 5, $|f| = 1 + 2 \cdot \phi + 2 \cdot \phi^2$



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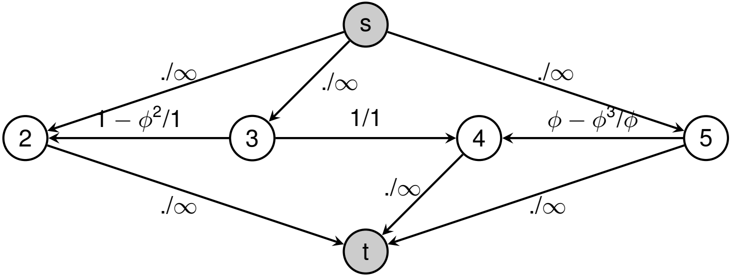

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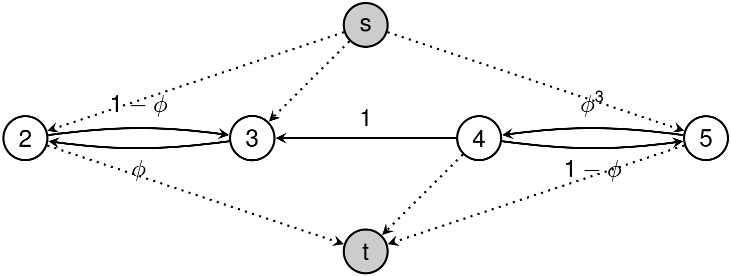

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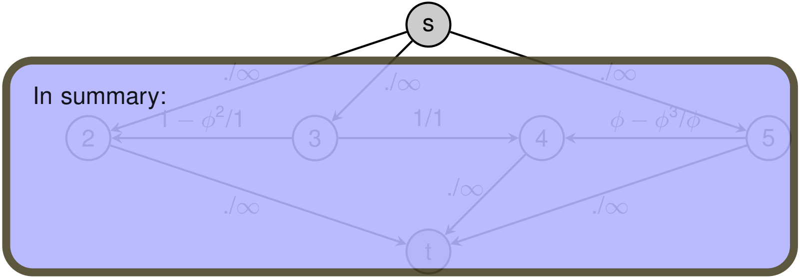
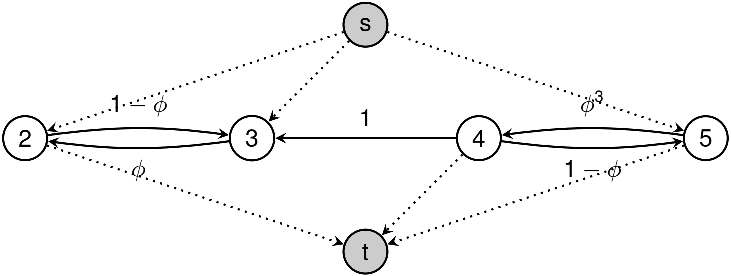
In summary:





Non-Termination of Ford-Fulkerson for Irrational Capacities

In summary:

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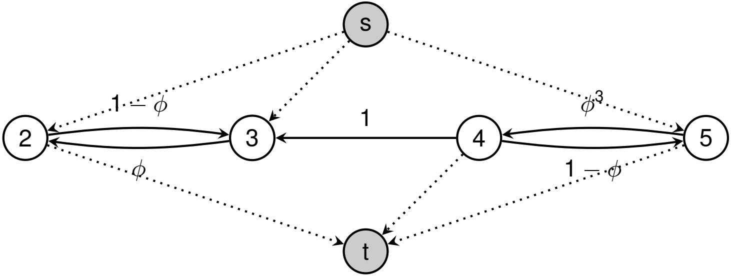

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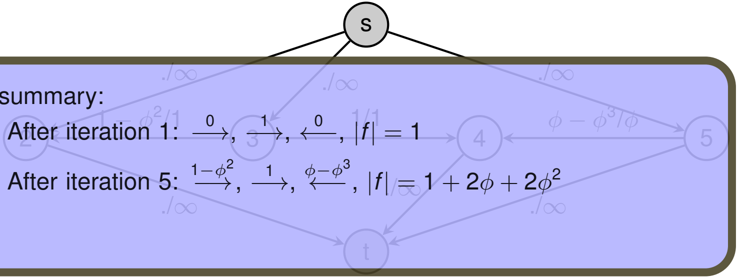

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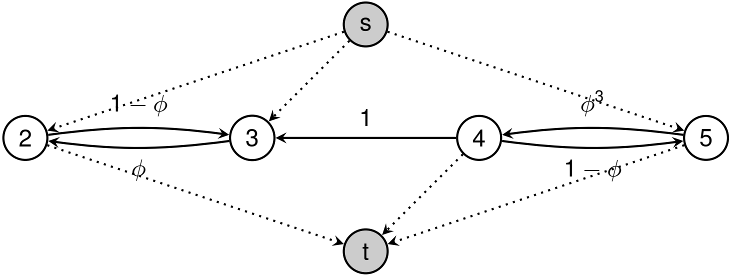

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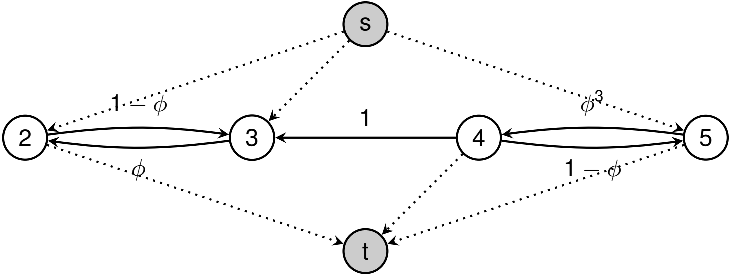

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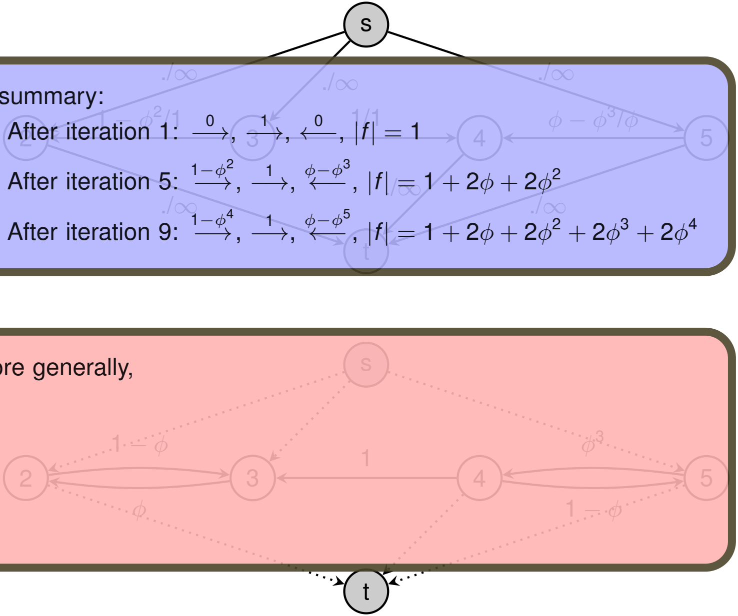

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More generally,

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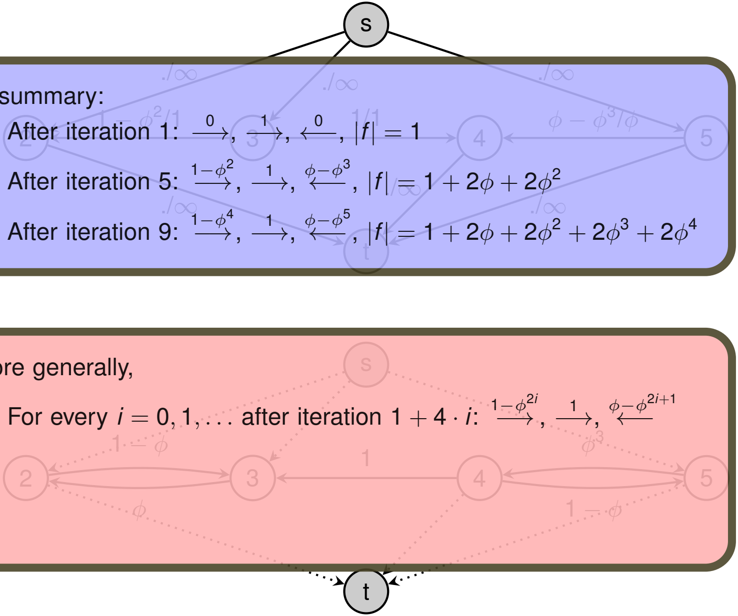

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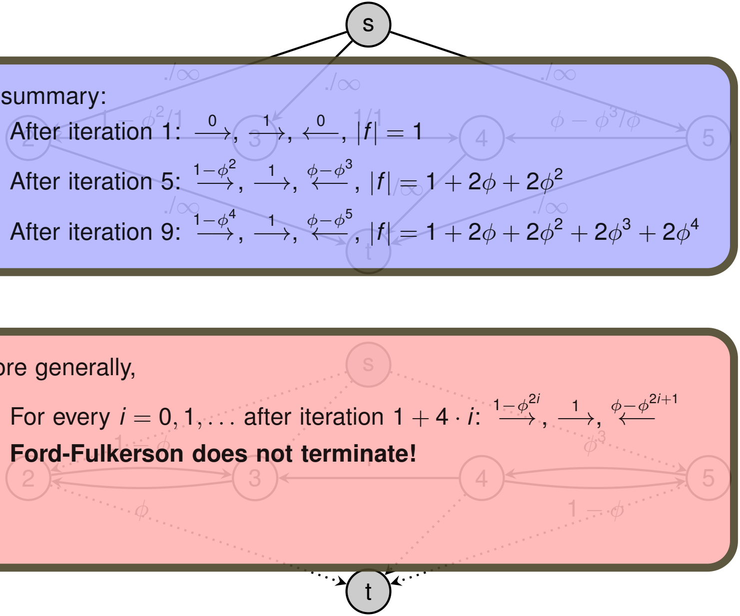
More generally,

- For every $i = 0, 1, \ldots$ after iteration $1 + 4 \cdot i$: $\xrightarrow{1-\phi^{2i}}$, $\xrightarrow{1}$, $\xleftarrow{\phi-\phi^{2i+1}}$

Non-Termination of Ford-Fulkerson for Irrational Capacities



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- **Ford-Fulkerson does not terminate!**

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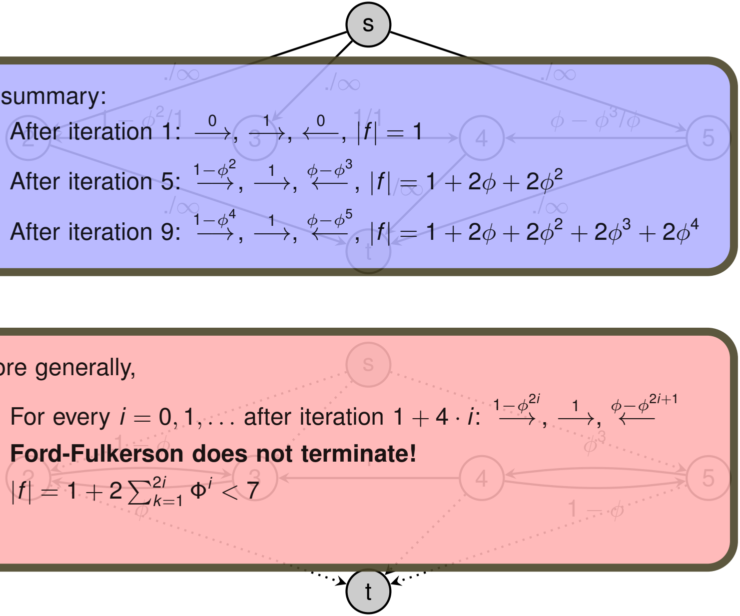

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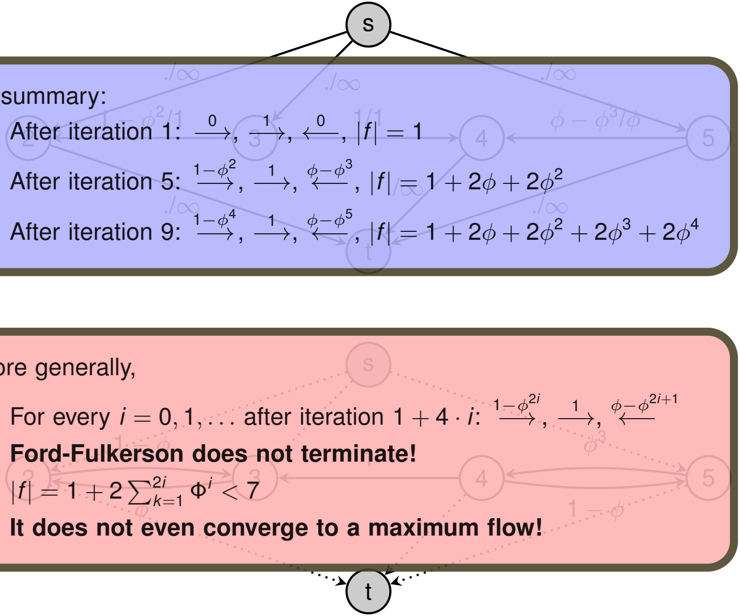

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More generally,

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- **Ford-Fulkerson does not terminate!**
- $|f| = 1 + 2\sum_{k=1}^{2i} \Phi^i < 7$
- **It does not even converge to a maximum flow!**

Summary and Outlook

Ford-Fulkerson Method

- works only for integral (rational) capacities

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- Runtime: $O(E \cdot |f^*|) = O(E \cdot V \cdot C)$

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Capacity-Scaling Algorithm

Summary and Outlook

Ford-Fulkerson Method

- works only for integral (rational) capacities
- Runtime: $O(E \cdot |f^*|) = O(E \cdot V \cdot C)$

Capacity-Scaling Algorithm

- Idea: Find an augmenting path with high capacity
- Consider subgraph of G_f consisting of edges (u, v) with $c_f(u, v) > \Delta$
- scaling parameter Δ, which is initially $2^{\lceil \log_2 C \rceil}$ and 1 after termination
- Runtime: $O(E^2 \cdot \log C)$

Summary and Outlook

Ford-Fulkerson Method

- works only for integral (rational) capacities
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Capacity-Scaling Algorithm

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Edmonds-Karp Algorithm

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Edmonds-Karp Algorithm

- Idea: Find the shortest augmenting path in G_f
- Runtime: $O(E^2 \cdot V)$