$$[\![\text{while } B \text{ do } C]\!] = \underline{fix}\left(f_{[\![B]\!],[\![C]\!]}\right)$$

Topic 2

Least Fixed Points

Thesis

All domains of computation are
partial orders with a least element.

Thesis

All domains of computation are partial orders with a least element.

All computable functions are mononotic.

Example: Every set can be made into a poset

$(S, \sqsubseteq)$ $\quad x \sqsubseteq y \; \forall x, y \in S$ iff$_{def}$ $x = y$

Partially ordered sets

A binary relation $\sqsubseteq$ on a set D is a partial order iff it is

reflexive: $\forall d \in D.\ d \sqsubseteq d$

transitive: $\forall d, d', d'' \in D.\ d \sqsubseteq d' \sqsubseteq d'' \Rightarrow d \sqsubseteq d''$

anti-symmetric: $\forall d, d' \in D.\ d \sqsubseteq d' \sqsubseteq d \Rightarrow d = d'$.

Such a pair $(D, \sqsubseteq)$ is called a partially ordered set, or poset.

$$\frac{}{x \sqsubseteq x}$$

$$\frac{x \sqsubseteq y \qquad y \sqsubseteq z}{x \sqsubseteq z}$$

$$\frac{x \sqsubseteq y \qquad y \sqsubseteq x}{x = y}$$

Domain of partial functions, $X \rightharpoonup Y$

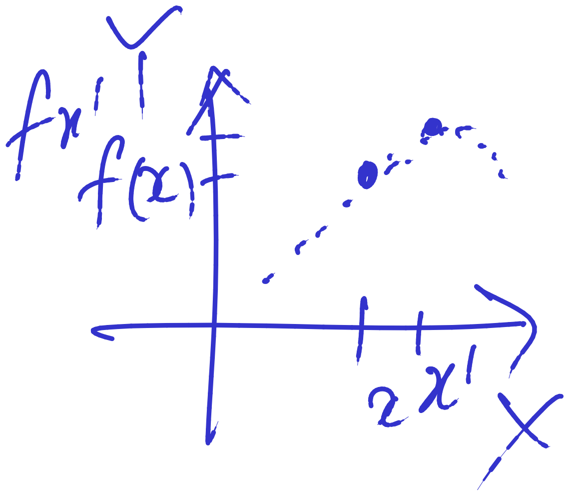
Domain of partial functions, $X \rightharpoonup Y$

Underlying set: all partial functions, f, with domain of definition $dom(f) \subseteq X$ and taking values in Y.

Def $graph(f) =_{df} \{ (x, f(x)) \in X \times Y \mid x \in dom(f) \}$

Domain of partial functions, $X \rightharpoonup Y$

Underlying set: all partial functions, f, with domain of definition $dom(f) \subseteq X$ and taking values in Y.



Partial order:

$$f \sqsubseteq g \quad \text{iff} \quad dom(f) \subseteq dom(g) \text{ and } \forall x \in dom(f).\ f(x) = g(x)$$

$$\text{iff} \quad graph(f) \subseteq graph(g)$$

We have $\perp =$ the totally undefined partial function (i.e. $graph(\perp) = \emptyset$) which is least $\perp \sqsubseteq f \ \forall f$.

Monotonicity

- A function $f : D \to E$ between posets is monotone iff

$$\forall d, d' \in D.\ d \sqsubseteq d' \Rightarrow f(d) \sqsubseteq f(d').$$

$$\frac{x \sqsubseteq y}{f(x) \sqsubseteq f(y)} \quad (f \text{ monotone})$$

Example: $S_\perp$ = underlying set $S \cup \{\perp\}$ ($\perp \notin S$) with order $x \sqsubseteq y$ $\forall x, y \in S_\perp$ iff $x = \perp$ or $x \neq \perp \Rightarrow x = y$

Least Elements

Suppose that D is a poset and that S is a subset of D.

An element $d \in S$ is the *least* element of S if it satisfies

$$\forall x \in S.\, d \sqsubseteq x \ .$$

Suppose both d and d' are least. Since d is least, $d \sqsubseteq d'$. Since d' is least, $d' \sqsubseteq d$. Hence $d = d'$.

- Note that because $\sqsubseteq$ is anti-symmetric, S has at most one least element.
- Note also that a poset may not have least element.

$S_\perp =$ 

Defined by so-called UNIVERSAL PROPERTIES.

Pre-fixed points

Let D be a poset and $f : D \to D$ be a function.

A fixed point is an element d s.t. $f(d) = d$

An element $d \in D$ is a pre-fixed point of f if it satisfies $f(d) \sqsubseteq d$.

The *least pre-fixed point* of f, if it exists, will be written

$$\boxed{\mathit{fix}(f)}$$

It is thus (uniquely) specified by the two properties:

$$f(\mathit{fix}(f)) \sqsubseteq \mathit{fix}(f) \qquad \text{(lfp1)}$$

$$\forall d \in D.\ f(d) \sqsubseteq d \ \Rightarrow\ \mathit{fix}(f) \sqsubseteq d. \qquad \text{(lfp2)}$$

fix(f) is a pre fixed point.

Amongst all pre fixed points we select fix(f) to be the least such.

Proof principle

$$\frac{f(x) \sqsubseteq x}{fix(f) \sqsubseteq x}$$

2. Let D be a poset and let $f : D \to D$ be a function with a least pre-fixed point $fix(f) \in D$.

Example

For all $x \in D$, to prove that $fix(f) \sqsubseteq x$ it is enough to establish that $f(x) \sqsubseteq x$.

$$\frac{f_{[\![B]\!],[\![C]\!]}\,[\![P]\!] \sqsubseteq [\![P]\!]}{fix(f_{[\![B]\!],[\![C]\!]}) = [\![\text{while } B \text{ do } C]\!] \sqsubseteq [\![P]\!]}$$

Proof principle

2. Let D be a poset and let $f : D \to D$ be a function with a least pre-fixed point $\mathit{fix}(f) \in D$.

   For all $x \in D$, to prove that $\mathit{fix}(f) \sqsubseteq x$ it is enough to establish that $f(x) \sqsubseteq x$.

$$\frac{f(x) \sqsubseteq x}{\mathit{fix}(f) \sqsubseteq x}$$

Proof principle

1.

$$\frac{}{f(\mathit{fix}(f)) \sqsubseteq \mathit{fix}(f)}$$

2. Let D be a poset and let $f : D \to D$ be a function with a least pre-fixed point $\mathit{fix}(f) \in D$.

   For all $x \in D$, to prove that $\mathit{fix}(f) \sqsubseteq x$ it is enough to establish that $f(x) \sqsubseteq x$.

$$\frac{f(x) \sqsubseteq x}{\mathit{fix}(f) \sqsubseteq x}$$

Lemma $f(\mathit{fix}\, f) = \mathit{fix}(f)$

Least pre-fixed points are fixed points

If it exists, the least pre-fixed point of a mononote function on a partial order is necessarily a fixed point. (lfp1)

$$
(a)\;\dfrac{(lfp1)\,\dfrac{}{f(\mathit{fix}\, f) \sqsubseteq \mathit{fix}\, f} \qquad (lfp2)\,\dfrac{(f\,mon)\,\dfrac{f(\mathit{fix}\, f) \sqsubseteq \mathit{fix}(f)}{f(f(\mathit{fix}\, f)) \sqsubseteq f(\mathit{fix}\, f)}}{\mathit{fix}\, f \sqsubseteq f(\mathit{fix}\, f)}}{f(\mathit{fix}\, f) = \mathit{fix}\, f}
$$

Remark: Not every monotone function on a poset has a least fixed point.

Example $\mathbb{B} = \{\text{true}, \text{false}\}$

$(\mathbb{B}, =)$ is a poset

Consider $\text{not} : \mathbb{B} \to \mathbb{B}$ is monotone but has no fixed points (at all).

Example $(\mathbb{N}, \leq)$ consider $\text{suc} : \mathbb{N} \to \mathbb{N}$

$\text{suc}(n) =_{\text{def}} n+1$

sums up all the information of the chain of d_i's $\leadsto \bigsqcup_n d_n$

Thesis*

All domains of computation are complete partial orders with a least element.

$$\begin{array}{c} \vdots \\ \sqcup\!\!\mid \\ d_n \\ \sqcup\!\!\mid \\ \vdots \\ \sqcup\!\!\mid \\ d_2 \\ \sqcup\!\!\mid \\ d_1 \\ \sqcup\!\!\mid \\ d_0 \end{array}$$

continuity

$\bigsqcup_n d_n = d \qquad \bigsqcup_n e_n = e \; /\!/ \; f(d)$

Thesis*

All domains of computation are
complete partial orders with a least element.

All computable functions are
continuous.

monotone + . . .

$d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \cdots \sqsubseteq d_n \sqsubseteq \cdots \sqsubseteq \bigsqcup_n d_n = d$

$e_0 = f(d_0) \sqsubseteq e_1 = f(d_1) \sqsubseteq \cdots \sqsubseteq e_n = f(d_n) \sqsubseteq \cdots \sqsubseteq \bigsqcup_n e_n = e$

$f(d_0) \sqsubseteq f(d_1) \sqsubseteq \cdots \sqsubseteq f(d_n) \sqsubseteq \cdots \sqsubseteq f(d)$

$D \xrightarrow{\quad f \quad} E$

idea sums up all the d_i's

Cpo's and domains

A chain complete poset, or cpo for short, is a poset $(D, \sqsubseteq)$ in which all countable increasing chains $d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \ldots$ have least upper bounds, $\bigsqcup_{n \geq 0} d_n$: → defined by univ. prop.

$$\forall m \geq 0 \,.\, d_m \sqsubseteq \bigsqcup_{n \geq 0} d_n \qquad \text{(lub1)}$$

$$\forall d \in D \,.\, (\forall m \geq 0 \,.\, d_m \sqsubseteq d) \Rightarrow \bigsqcup_{n \geq 0} d_n \sqsubseteq d. \qquad \text{(lub2)}$$

a b c

$\bigsqcup_n d_n$

d_n

d_1

d_0

A domain is a cpo that possesses a least element, $\perp$:

$$\forall d \in D \,.\, \perp \sqsubseteq d.$$

$$\frac{}{\bot \sqsubseteq x}$$

(Lub 1)

$$\frac{}{x_i \sqsubseteq \bigsqcup_{n \geq 0} x_n} \quad (i \geq 0 \text{ and } \langle x_n \rangle \text{ a chain})$$

(lub2)

$$\frac{\forall n \geq 0 \,.\, x_n \sqsubseteq x}{\bigsqcup_{n \geq 0} x_n \sqsubseteq x} \quad (\langle x_i \rangle \text{ a chain})$$

Domain of partial functions, $X \rightharpoonup Y$

Domain of partial functions, $X \rightharpoonup Y$

Underlying set: all partial functions, f, with domain of definition $dom(f) \subseteq X$ and taking values in Y.

Domain of partial functions, $X \rightharpoonup Y$

Underlying set: all partial functions, f, with domain of definition $dom(f) \subseteq X$ and taking values in Y.

Partial order:

$$
\begin{array}{lll}
f \sqsubseteq g & \text{iff} & dom(f) \subseteq dom(g) \text{ and} \\
 & & \forall x \in dom(f).\ f(x) = g(x) \\
 & \text{iff} & graph(f) \subseteq graph(g)
\end{array}
$$

Domain of partial functions, $X \rightharpoonup Y$

Underlying set: all partial functions, f, with domain of definition $dom(f) \subseteq X$ and taking values in Y.

Partial order:

$$f \sqsubseteq g \quad \text{iff} \quad dom(f) \subseteq dom(g) \text{ and } \forall x \in dom(f).\ f(x) = g(x)$$

$$\text{iff} \quad graph(f) \subseteq graph(g)$$

Lub of chain $f_0 \sqsubseteq f_1 \sqsubseteq f_2 \sqsubseteq \ldots$ is the partial function f with $dom(f) = \bigcup_{n \geq 0} dom(f_n)$ and

$$f(x) = \begin{cases} f_n(x) & \text{if } x \in dom(f_n), \text{ some } n \\ \text{undefined} & \text{otherwise} \end{cases}$$

In fact, $\underline{graph}(f) = \bigcup_n graph$

Domain of partial functions, $X \rightharpoonup Y$

Underlying set: all partial functions, f, with domain of definition $dom(f) \subseteq X$ and taking values in Y.

Partial order:

$$f \sqsubseteq g \quad \text{iff} \quad dom(f) \subseteq dom(g) \text{ and } \forall x \in dom(f).\ f(x) = g(x)$$

$$\text{iff} \quad graph(f) \subseteq graph(g)$$

Lub of chain $f_0 \sqsubseteq f_1 \sqsubseteq f_2 \sqsubseteq \ldots$ is the partial function f with $dom(f) = \bigcup_{n \geq 0} dom(f_n)$ and

$$f(x) = \begin{cases} f_n(x) & \text{if } x \in dom(f_n), \text{ some } n \\ \text{undefined} & \text{otherwise} \end{cases}$$

Least element $\perp$ is the totally undefined partial function.