

Domain of partial functions, $X \rightharpoonup Y$

Underlying set: all partial functions, f , with domain of definition $\text{dom}(f) \subseteq X$ and taking values in Y .

Partial order:

$$\begin{aligned} f \sqsubseteq g &\quad \text{iff} \quad \text{dom}(f) \subseteq \text{dom}(g) \text{ and} \\ &\quad \forall x \in \text{dom}(f). f(x) = g(x) \\ &\quad \text{iff} \quad \text{graph}(f) \subseteq \text{graph}(g) \end{aligned}$$

Lub of chain $f_0 \sqsubseteq f_1 \sqsubseteq f_2 \sqsubseteq \dots$ is the partial function f with $\text{dom}(f) = \bigcup_{n \geq 0} \text{dom}(f_n)$ and

$$f(x) = \begin{cases} f_n(x) & \text{if } x \in \text{dom}(f_n), \text{ some } n \\ \text{undefined} & \text{otherwise} \end{cases}$$

is a
domain

Least element \perp is the totally undefined partial function.

Remark
 $(X \rightarrow \{\ast\})$
 $\cong P(X)$

The domain of subsets of a set x : $P(x)$

$(P(x), \subseteq)$

• $\emptyset \subseteq S \forall S \in P(x)$ hence it is least

• lubs of ω -chains

$$S_0 \subseteq S_1 \subseteq \dots \subseteq S_n \subseteq \dots \quad (n \in \omega)$$

has lub $\bigcup_{n \in \omega} S_n$

1. In the constantly d chain; i.e. $d \sqsubseteq d \sqsubseteq \dots \sqsubseteq d \sqsubseteq \dots$
 we have $\bigsqcup_n d = d$

Some properties of lubs of chains

Let D be a cpo.

1. For $d \in D$, $\bigsqcup_n d = d$.

2. For every chain $d_0 \sqsubseteq d_1 \sqsubseteq \dots \sqsubseteq d_n \sqsubseteq \dots$ in D ,

$$\bigsqcup_n d_n = \bigsqcup_n d_{N+n}$$

for all $N \in \mathbb{N}$.

$$\bigsqcup_n d_n = \bigsqcup_R^L d_{N+R}^L$$

The diagram illustrates the proof of property 2. It shows two chains of elements from a partially ordered set D . The left chain consists of elements d_0, d_1, \dots, d_N , with d_0 at the bottom and d_N at the top. The right chain consists of elements $d_{N+1}, d_{N+2}, \dots, d_{N+R}$, also with d_{N+1} at the bottom and d_{N+R} at the top. Above each chain, a series of upward arrows indicates the partial order: $d_0 \sqsubseteq d_1 \sqsubseteq \dots \sqsubseteq d_N$ on the left, and $d_{N+1} \sqsubseteq d_{N+2} \sqsubseteq \dots \sqsubseteq d_{N+R}$ on the right. To the right of the chains, a single upward arrow labeled \sqsubseteq connects the top element d_N to the top element d_{N+R} , representing the lub of the original chain being equal to the lub of the extended chain.

$$x_i \in \bigcup_j x_n \cdot h_i$$

$$d_{N+i} \in \bigcup_k d_{N+k} \cdot h_i$$

$$d_1? \quad d_2? \quad \dots \quad d_{N-1}?$$

$$d_0 \in d_N \in \bigcup_k d_{N+k}$$

$$f(n) \cdot d_n \in \bigcup_k d_{N+k}$$

$$\bigcup_n d_n \in \bigcup_k d_{N+k}$$

apply
same
reasoning

$$\bigvee_k d_{N+k} \in \bigcup_n d_n$$

$$\bigcup_k d_{N+k} \subseteq \bigcup_n d_n$$

$$\bigcup_n d_n = \bigcup_k d_{N+k}$$

$$\bigcup_k d_k \subseteq \bigcup_l d_{n+l}$$

3. For every pair of chains $d_0 \sqsubseteq d_1 \sqsubseteq \dots \sqsubseteq d_n \sqsubseteq \dots$ and $e_0 \sqsubseteq e_1 \sqsubseteq \dots \sqsubseteq e_n \sqsubseteq \dots$ in D ,

if $d_n \sqsubseteq e_n$ for all $n \in \mathbb{N}$ then $\bigcup_n d_n \sqsubseteq \bigcup_n e_n$.

$$\frac{\begin{array}{c} d_n \sqsubseteq e_n \\ e_n \sqsubseteq \bigcup_m e_m \end{array}}{\forall n \quad d_n \sqsubseteq \bigcup_m e_m} \quad \frac{}{\bigcup_n d_n \sqsubseteq \bigcup_m e_m}$$

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if $d_n \sqsubseteq e_n$ for all $n \in \mathbb{N}$ then $\bigsqcup_n d_n \sqsubseteq \bigsqcup_n e_n$.

$$\frac{\forall n \geq 0 . x_n \sqsubseteq y_n}{\bigsqcup_n x_n \sqsubseteq \bigsqcup_n y_n} \quad (\langle x_n \rangle \text{ and } \langle y_n \rangle \text{ chains})$$

$$\bigcup_n d_n^{(0)} \subseteq \bigcup_n d_n^{(1)} \subseteq \dots$$

$$\begin{matrix} & \vdots \\ u_1 & \\ d_n^{(0)} & \subseteq \\ u_1 & \end{matrix} \quad \begin{matrix} & \vdots \\ u_1 & \\ d_n^{(1)} & \subseteq \\ u_1 & \end{matrix} \quad \begin{matrix} & \vdots \\ u_1 & \\ d_n^{(2)} & \subseteq \\ u_1 & \end{matrix} \quad \dots \quad \subseteq$$

$$\begin{matrix} & \vdots \\ u_1 & \\ d_2^{(0)} & \subseteq \\ u_1 & \end{matrix} \quad \begin{matrix} u_1 \\ d_2^{(1)} & \subseteq \\ u_1 & \end{matrix} \quad \begin{matrix} u_1 \\ d_2^{(2)} & \subseteq \\ u_1 & \end{matrix} \quad \dots$$

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$$\begin{matrix} & \vdots \\ u_1 & \\ d_0^{(0)} & \subseteq \\ u_1 & \end{matrix} \quad \begin{matrix} u_1 \\ d_0^{(1)} & \subseteq \\ u_1 & \end{matrix} \quad \begin{matrix} u_1 \\ d_0^{(2)} & \subseteq \\ u_1 & \end{matrix} \quad \dots \subseteq d_0^{(m)} \subseteq \dots$$

$$\bigcup_m \left(\bigcup_n d_n^{(m)} \right)$$

$$\begin{matrix} & \vdots \\ u_1 & \\ d_R^{(R)} & \subseteq \\ u_1 & \end{matrix}$$

$$\bigcup_m d_n^m$$

$$\begin{matrix} & \vdots \\ u_1 & \\ d_2^{(m)} & \subseteq \\ u_1 & \end{matrix} \quad \dots$$

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$$\begin{matrix} & \vdots \\ u_1 & \\ \bigcup_m d_1^{(m)} & \subseteq \\ u_1 & \end{matrix}$$

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$$x_i \in \bigcup_j x_j$$

$$d_n^{(m)} \in \bigcup_n d_n^{(m)}$$

$$d_n^{(m)} \subseteq \bigcup_m d_n^{(m)} \subseteq \bigcup_m \bigcup_n d_n^{(m)} \quad (\times)$$

$$\text{(*)} \quad \frac{x_n \subseteq y_n}{\bigcup_n x_n \subseteq \bigcup_n y_n}$$

$$d_n^{(m)} \subseteq d_{\max(m,n)}^{(\max(m,n))} \subseteq \bigcup_R d_R^{(R)}$$

$$\forall n \quad d_n^{(m)} \subseteq \bigcup_R d_R^{(R)}$$

$$\forall m \quad \bigcup_n d_n^{(m)} \subseteq \bigcup_R d_R^{(R)}$$

$$\bigcup_k d_k^{(R)} \subseteq \bigcup_m \bigcup_n d_n^{(m)}$$

$$\bigcup_m \bigcup_n d_n^{(m)} \subseteq \bigcup_R d_R^{(R)}$$

$$\bigcup_k d_k^{(R)} = \bigcup_m \bigcup_n d_n^{(m)}$$

Diagonalising a double chain

Lemma. Let D be a cpo. Suppose that the doubly-indexed family of elements $d_{m,n} \in D$ ($m, n \geq 0$) satisfies

$$m \leq m' \ \& \ n \leq n' \Rightarrow d_{m,n} \sqsubseteq d_{m',n'}. \quad (\dagger)$$

Then

$$\bigsqcup_{n \geq 0} d_{0,n} \sqsubseteq \bigsqcup_{n \geq 0} d_{1,n} \sqsubseteq \bigsqcup_{n \geq 0} d_{2,n} \sqsubseteq \dots$$

and

$$\bigsqcup_{m \geq 0} d_{m,0} \sqsubseteq \bigsqcup_{m \geq 0} d_{m,1} \sqsubseteq \bigsqcup_{m \geq 0} d_{m,3} \sqsubseteq \dots$$

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$$\bigsqcup_{m \geq 0} d_{m,0} \sqsubseteq \bigsqcup_{m \geq 0} d_{m,1} \sqsubseteq \bigsqcup_{m \geq 0} d_{m,3} \sqsubseteq \dots$$

Moreover

$$\bigsqcup_{m \geq 0} \left(\bigsqcup_{n \geq 0} d_{m,n} \right) = \bigsqcup_{k \geq 0} d_{k,k} = \bigsqcup_{n \geq 0} \left(\bigsqcup_{m \geq 0} d_{m,n} \right).$$

$$\bigcup_n d_n \xrightarrow{\text{upper bound}} f\left(\bigcup_n d_n\right) = \bigcup_n f(d_n)$$

Continuity and strictness

- If D and E are cpo's, the function f is **continuous** iff
 1. it is monotone, and
 2. it preserves lubs of chains, i.e. for all chains $d_0 \sqsubseteq d_1 \sqsubseteq \dots$ in D , it is the case that

$$f\left(\bigcup_{n \geq 0} d_n\right) = \bigcup_{n \geq 0} f(d_n) \text{ in } E.$$

$$D \xrightarrow{f} E$$

Continuity and strictness

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preserves
|| least

- If D and E have least elements, then the function f is **strict** iff $f(\perp) = \perp$.
Example: $\text{id}_D : D \rightarrow D$ strict } element.

$(\perp \neq d \in D)$

$\lambda x.d : D \rightarrow D$ non strict

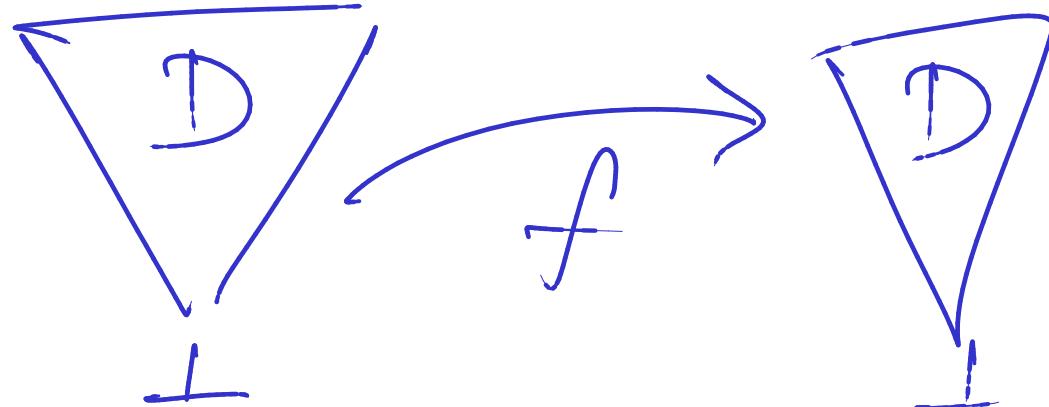
Tarski's Fixed Point Theorem

Let $f : D \rightarrow D$ be a continuous function on a domain D . Then

- f possesses a least pre-fixed point, given by

$$\text{fix}(f) = \bigsqcup_{n \geq 0} f^n(\perp).$$

- Moreover, $\text{fix}(f)$ is a fixed point of f , i.e. satisfies $f(\text{fix}(f)) = \text{fix}(f)$, and hence is the **least fixed point** of f .



$\text{fix}(f)$ $\in D$ s.t. $f(\underline{\text{fix}(f)}) = \underline{\text{fix}(f)}$

{ least pre-fixed point \nearrow

$\perp \leq f(\perp) \Rightarrow f(\perp) \leq f f(\perp) \Rightarrow f f(\perp) \leq f f f(\perp) \Rightarrow \dots$

$\perp \leq f(\perp) \leq f f(\perp) \leq \dots \leq f^n(\perp) \leq \dots \bigcup_n f^n(\perp)$

CLAIM " $\text{fix}(f)$ "