Domain of partial functions, $X \rightharpoonup Y$

Underlying set: all partial functions, f, with domain of definition $dom(f) \subseteq X$ and taking values in Y.

Remark

$(X \rightharpoonup \{*\}) \cong \mathcal{P}(X)$

Partial order:

$$f \sqsubseteq g \quad \text{iff} \quad dom(f) \subseteq dom(g) \text{ and } \forall x \in dom(f).\ f(x) = g(x)$$

$$\text{iff} \quad graph(f) \subseteq graph(g)$$

is a domain

Lub of chain $f_0 \sqsubseteq f_1 \sqsubseteq f_2 \sqsubseteq \ldots$ is the partial function f with $dom(f) = \bigcup_{n \geq 0} dom(f_n)$ and

$$f(x) = \begin{cases} f_n(x) & \text{if } x \in dom(f_n), \text{ some } n \\ \text{undefined} & \text{otherwise} \end{cases}$$

Least element $\bot$ is the totally undefined partial function.

The domain of subsets of a set X : $\mathcal{P}(X)$

$(\mathcal{P}(X), \subseteq)$

- $\emptyset \subseteq S \ \forall S \in \mathcal{P}(X)$ hence it is least
- lubs of ω-chains

$$S_0 \subseteq S_1 \subseteq \dots \subseteq S_n \subseteq \dots \quad (n \in \mathbb{N})$$

has lub $\bigcup_{n \in \mathbb{N}} S_n$

1. For the constantly d chain; i.e $d \sqsubseteq d \sqsubseteq \dots \sqsubseteq d \sqsubseteq \dots$ we have $\bigsqcup_n d = d$

$$\bigsqcup_n d_n = \bigsqcup_k d_{N+k}$$

Some properties of lubs of chains

$d_0 \sqsubseteq d_1 \sqsubseteq \dots \sqsubseteq d_n \sqsubseteq \dots$

$d_N \sqsubseteq d_{N+1} \sqsubseteq \dots$

Let D be a cpo.

1. For $d \in D$, $\bigsqcup_n d = d$.
2. For every chain $d_0 \sqsubseteq d_1 \sqsubseteq \dots \sqsubseteq d_n \sqsubseteq \dots$ in D,

$$\bigsqcup_n d_n = \bigsqcup_n d_{N+n}$$

for all $N \in \mathbb{N}$.

$x_i \sqsubseteq \bigsqcup_j x_n \quad \forall i$

$d_{N+i} \sqsubseteq \bigsqcup_k d_{N+k} \quad \forall i$

$d_1 ? \quad d_2 ? \quad \cdots \quad d_{N-1} ?$

$d_0 \sqsubseteq d_N \sqsubseteq \bigsqcup_k d_{N+k}$

apply same reasoning

$$\frac{\forall n \quad d_n \sqsubseteq \bigsqcup_k d_{N+k}}{\bigsqcup_n d_n \sqsubseteq \bigsqcup_k d_{N+k}}$$

$$\frac{\forall k.\ d_{N+k} \sqsubseteq \bigsqcup_n d_n}{\bigsqcup_k d_{N+k} \sqsubseteq \bigsqcup_n d_n}$$

$$\bigsqcup_n d_n = \bigsqcup_k d_{N+k}$$

$$\bigsqcup_k d_k \sqsubseteq \bigsqcup_\ell d_{N+\ell}$$

$$\bigsqcup_n d_n \sqsubseteq \bigsqcup_n e_n$$

3. For every pair of chains $d_0 \sqsubseteq d_1 \sqsubseteq \ldots \sqsubseteq d_n \sqsubseteq \ldots$ and $e_0 \sqsubseteq e_1 \sqsubseteq \ldots \sqsubseteq e_n \sqsubseteq \ldots$ in D,

if $d_n \sqsubseteq e_n$ for all $n \in \mathbb{N}$ then $\bigsqcup_n d_n \sqsubseteq \bigsqcup_n e_n$.

$$\cfrac{\cfrac{d_n \sqsubseteq e_n \qquad e_n \sqsubseteq \bigsqcup_m e_m}{\forall n \;\; d_n \sqsubseteq \bigsqcup_m e_m}}{\bigsqcup_n d_n \sqsubseteq \bigsqcup_m e_m}$$

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$$\frac{\forall n \geq 0 \,.\, x_n \sqsubseteq y_n}{\bigsqcup_n x_n \sqsubseteq \bigsqcup_n y_n} \quad (\langle x_n \rangle \text{ and } \langle y_n \rangle \text{ chains})$$

$$
\begin{array}{cccccccccccc}
\bigsqcup_n d_n^{(0)} & \sqsubseteq & \bigsqcup_n d_n^{(1)} & \sqsubseteq & \cdots & & & & \bigsqcup_m\left(\bigsqcup_n d_n^{(m)}\right) & & & \bigsqcup_n\left(\bigsqcup_m d_n^{(m)}\right) \\
 & & & & & & & & & \| & & \\
\vdots & & \vdots & & \vdots & & & & \vdots & \bigsqcup_k d_k^{(k)} & = & \vdots \\
\rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & & & \rotatebox{90}{$\sqsubseteq$} & & & \rotatebox{90}{$\sqsubseteq$} \\
d_n^{(0)} & \sqsubseteq & d_n^{(1)} & \sqsubseteq & d_n^{(2)} & \sqsubseteq & \cdots & \sqsubseteq & d_n^{(m)} & \sqsubseteq & \cdots & \bigsqcup_m d_n^{m} \\
\rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & & \rotatebox{45}{$\sqsubseteq$} & \rotatebox{90}{$\sqsubseteq$} & & & \rotatebox{90}{$\sqsubseteq$} \\
 & & \vdots & & \vdots & & \cdots & & \vdots & & & \\
\vdots & & \rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & \rotatebox{45}{$\sqsubseteq$} & & & \rotatebox{90}{$\sqsubseteq$} & & & \vdots \\
\rotatebox{90}{$\sqsubseteq$} & & & & & & & & & & & \\
d_2^{(0)} & \sqsubseteq & d_2^{(1)} & \sqsubseteq & d_2^{(2)} & \sqsubseteq & \cdots & & d_2^{(m)} & \sqsubseteq & \cdots & \\
 & & \rotatebox{90}{$\sqsubseteq$} & \rotatebox{45}{$\sqsubseteq$} & \rotatebox{90}{$\sqsubseteq$} & & & & \rotatebox{90}{$\sqsubseteq$} & & & \rotatebox{90}{$\sqsubseteq$} \\
\rotatebox{90}{$\sqsubseteq$} & & & & & & & & & & & \\
d_1^{(0)} & \sqsubseteq & d_1^{(1)} & \sqsubseteq & d_1^{(2)} & \sqsubseteq & \cdots & & d_1^{(m)} & \sqsubseteq & \cdots & \bigsqcup_m d_1^{(m)} \\
\rotatebox{90}{$\sqsubseteq$} & \rotatebox{45}{$\sqsubseteq$} & \rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & & & \rotatebox{90}{$\sqsubseteq$} & & & \rotatebox{90}{$\sqsubseteq$} \\
d_0^{(0)} & \sqsubseteq & d_0^{(1)} & \sqsubseteq & d_0^{(2)} & \sqsubseteq & \cdots & \sqsubseteq & d_0^{(m)} & \sqsubseteq & \cdots & \bigsqcup_m d_0^{(m)}
\end{array}
$$

$$\frac{}{x_i \sqsubseteq \bigsqcup_j x_j}$$

$$(*)\quad \frac{x_n \sqsubseteq y_n}{\bigsqcup_n x_n \sqsubseteq \bigsqcup_n y_n}$$

$$\frac{d_n^{(m)} \sqsubseteq \bigsqcup_n d_n^{(m)}}{d_n^{(m)} \sqsubseteq \bigsqcup_m d_n^{(m)} \sqsubseteq \bigsqcup_m \bigsqcup_n d_n^{(m)}}\,(*)$$

$$\dfrac{\dfrac{\dfrac{d_n^{(m)} \sqsubseteq d_{\max(m,n)}^{(\max(m,n))} \sqsubseteq \bigsqcup_k d_k^{(k)}}{\forall m\, \forall n\quad d_n^{(m)} \sqsubseteq \bigsqcup_k d_k^{(k)}}}{\forall m\quad \bigsqcup_n d_n^{(m)} \sqsubseteq \bigsqcup_k d_k^{(k)}}}{\bigsqcup_m \bigsqcup_n d_n^{(m)} \sqsubseteq \bigsqcup_k d_k^{(k)}}$$

$$\frac{\forall k\quad d_k^{(k)} \sqsubseteq \bigsqcup_m \bigsqcup_n d_n^{(m)}}{\bigsqcup_k d_k^{(k)} \sqsubseteq \bigsqcup_m \bigsqcup_n d_n^{(m)}}$$

$$\bigsqcup_k d_k^{(k)} = \bigsqcup_m \bigsqcup_n d_n^{(m)}$$

Diagonalising a double chain

Lemma. *Let D be a cpo. Suppose that the doubly-indexed family of elements $d_{m,n} \in D$ ($m, n \geq 0$) satisfies*

$$m \leq m' \;\&\; n \leq n' \;\Rightarrow\; d_{m,n} \sqsubseteq d_{m',n'}. \qquad (\dagger)$$

Then

$$\bigsqcup_{n\geq 0} d_{0,n} \;\sqsubseteq\; \bigsqcup_{n\geq 0} d_{1,n} \;\sqsubseteq\; \bigsqcup_{n\geq 0} d_{2,n} \;\sqsubseteq\; \ldots$$

and

$$\bigsqcup_{m\geq 0} d_{m,0} \;\sqsubseteq\; \bigsqcup_{m\geq 0} d_{m,1} \;\sqsubseteq\; \bigsqcup_{m\geq 0} d_{m,3} \;\sqsubseteq\; \ldots$$

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Moreover

$$\bigsqcup_{m \geq 0} \left(\bigsqcup_{n \geq 0} d_{m,n} \right) = \bigsqcup_{k \geq 0} d_{k,k} = \bigsqcup_{n \geq 0} \left(\bigsqcup_{m \geq 0} d_{m,n} \right) .$$

Continuity and strictness

- If D and E are cpo's, the function f is continuous iff
  1. it is monotone, and
  2. it preserves lubs of chains, *i.e.* for all chains $d_0 \sqsubseteq d_1 \sqsubseteq \ldots$ in D, it is the case that

$$f(\bigsqcup_{n\geq 0} d_n) = \bigsqcup_{n\geq 0} f(d_n) \quad \text{in } E.$$

$\bigsqcup_n d_n \mapsto f(\bigsqcup_n d_n) \sqsupseteq \bigsqcup_n f(d_n)$

upper bound

$\Leftarrow$ continuity

$d_0 \sqsubseteq d_1 \sqsubseteq \ldots \sqsubseteq d_n \sqsubseteq \ldots \sqsubseteq \bigsqcup_n d_n$ D

$f(d_0) \sqsubseteq f(d_1) \sqsubseteq f(d_2) \sqsubseteq \ldots$ E

$D \xrightarrow{f} E$

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preserves
|| least
element.

- If D and E have least elements, then the function f is strict iff $f(\bot) = \bot$.

Example: $id_D : D \to D$ strict

$(\bot \neq d \in D)$ $\lambda x.d : D \to D$ nonstrict

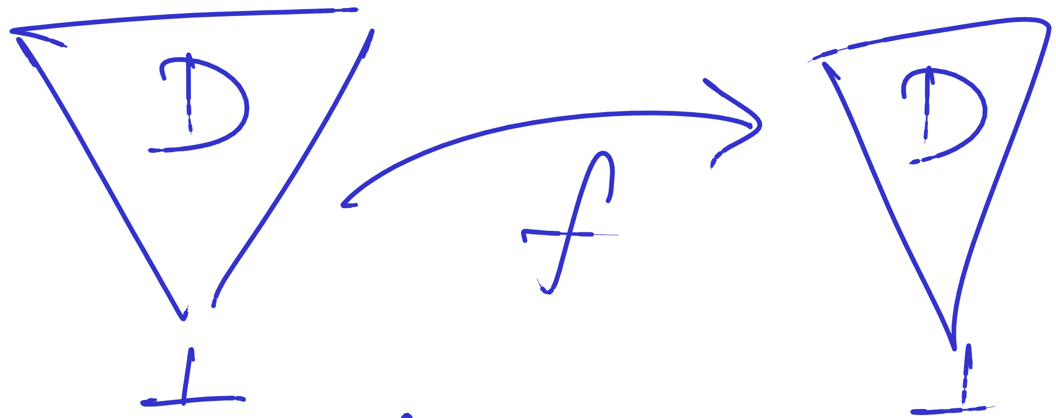
Tarski's Fixed Point Theorem

Let $f : D \to D$ be a continuous function on a domain D. Then

- f possesses a least pre-fixed point, given by

$$\mathit{fix}(f) = \bigsqcup_{n \geq 0} f^n(\bot).$$

- Moreover, $\mathit{fix}(f)$ is a fixed point of f, *i.e.* satisfies $f\big(\mathit{fix}(f)\big) = \mathit{fix}(f)$, and hence is the least fixed point of f.



$\underbrace{fix(f)}_{\text{least pre-fixed point}} \in D$ s.t. $f(fix(f)) = fix(f)$

$$\bot \sqsubseteq f(\bot) \overset{\text{mon}}{\Longrightarrow} f(\bot) \sqsubseteq ff(\bot) \Longrightarrow ff(\bot) \sqsubseteq fff(\bot) \Longrightarrow \cdots$$

$$\bot \sqsubseteq f(\bot) \sqsubseteq ff(\bot) \sqsubseteq \cdots \sqsubseteq f^n(\bot) \sqsubseteq \cdots \bigsqcup_n f^n(\bot)$$

CLAIM $\bigsqcup_n f^n(\bot) = fix(f)$