PCF evaluation relation

takes the form

$$M \Downarrow_{\tau} V$$

where

- τ is a PCF type
- $M, V \in \mathrm{PCF}_{\tau}$ are closed PCF terms of type τ
- V is a value,

$$V ::= \mathbf{0} \mid \mathbf{succ}(V) \mid \mathbf{true} \mid \mathbf{false} \mid \mathbf{fn}\, x : \tau \,.\, M.$$

PCF evaluation (sample rules)

$(\Downarrow_{\mathrm{val}})$ $\quad V \Downarrow_\tau V$ $\qquad$ (V a value of type τ)

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$(\Downarrow_{\text{val}}) \quad V \Downarrow_\tau V \qquad$ (V a value of type τ)

$$(\Downarrow_{\text{cbn}}) \quad \frac{M_1 \Downarrow_{\tau \to \tau'} \mathbf{fn}\, x : \tau \,.\, M_1' \qquad M_1'[M_2/x] \Downarrow_{\tau'} V}{M_1\, M_2 \Downarrow_{\tau'} V}$$

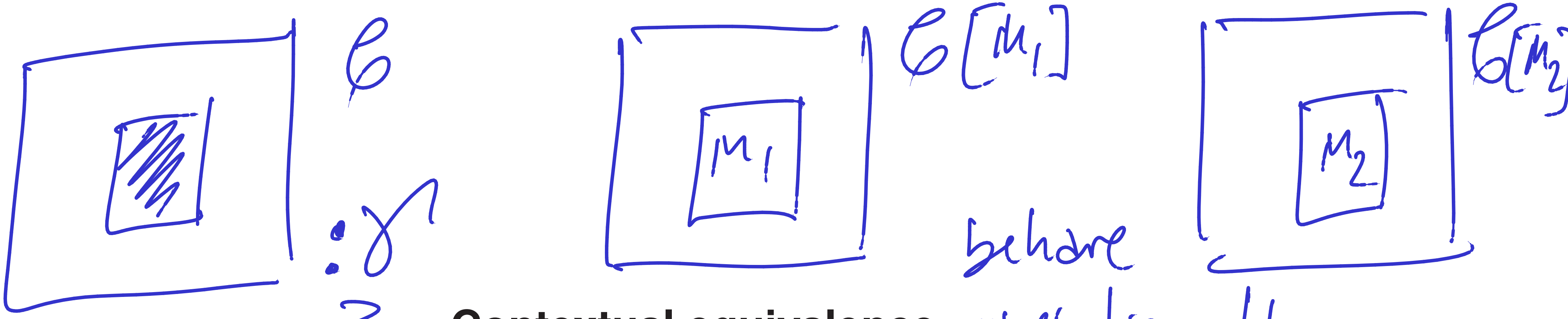
Call-by-name

PCF evaluation (sample rules)

$$(\Downarrow_{\mathrm{val}}) \quad V \Downarrow_\tau V \qquad (V \text{ a value of type } \tau)$$

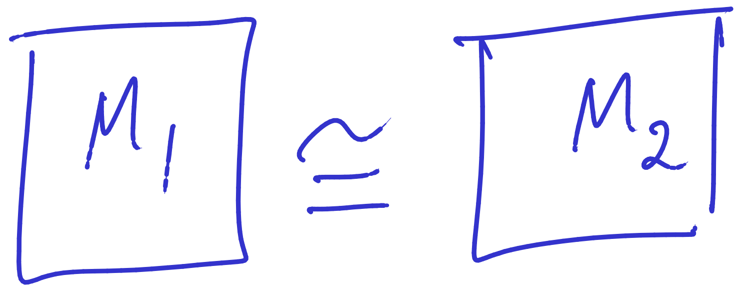
$$(\Downarrow_{\mathrm{cbn}}) \quad \frac{M_1 \Downarrow_{\tau \to \tau'} \mathbf{fn}\, x : \tau \,.\, M_1' \qquad M_1'[M_2/x] \Downarrow_{\tau'} V}{M_1\, M_2 \Downarrow_{\tau'} V}$$

$$(\Downarrow_{\mathrm{fix}}) \quad \frac{M\big(\mathbf{fix}(M)\big) \Downarrow_\tau V}{\mathbf{fix}(M) \Downarrow_\tau V}$$



Contextual equivalence

Two phrases of a programming language are contextually equivalent if any occurrences of the first phrase in a complete program can be replaced by the second phrase without affecting the observable results of executing the program.



Contextual equivalence of PCF terms

Given PCF terms M_1, M_2, PCF type τ, and a type environment Γ, the relation $\boxed{\Gamma \vdash M_1 \cong_{\mathrm{ctx}} M_2 : \tau}$ is defined to hold iff

- Both the typings $\Gamma \vdash M_1 : \tau$ and $\Gamma \vdash M_2 : \tau$ hold.
- For all PCF contexts $\mathcal{C}$ for which $\mathcal{C}[M_1]$ and $\mathcal{C}[M_2]$ are closed terms of type γ, *where* $\gamma = nat$ *or* $\gamma = bool$, and for all values $V : \gamma$,

$$\mathcal{C}[M_1] \Downarrow_\gamma V \;\Leftrightarrow\; \mathcal{C}[M_2] \Downarrow_\gamma V.$$

PCF denotational semantics — aims

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- PCF types $\tau \mapsto$ domains $[\![\tau]\!]$.

$$[\![nat]\!] \stackrel{def}{=} \mathbb{N}_\perp$$

$$[\![bool]\!] \stackrel{def}{=} \mathbb{B}_\perp$$

$$[\![\sigma \to \tau]\!] \stackrel{def}{=} ([\![\sigma]\!] \to [\![\tau]\!])$$

PCF denotational semantics — aims

- PCF types $\tau \;\mapsto\;$ domains $[\![\tau]\!]$.
- Closed PCF terms $M : \tau \;\mapsto\;$ elements $[\![M]\!] \in [\![\tau]\!]$.

  Denotations of open terms will be continuous functions.

$$[\![\Gamma \vdash M : \tau]\!] : [\![\Gamma]\!] \longrightarrow [\![\tau]\!] \quad \text{continuous}$$

domain interpreting the environment Γ

(ex: $[\![x : \sigma \vdash M : \tau]\!] : [\![\sigma]\!] \xrightarrow[\text{cont}]{} [\![\tau]\!]$)

PCF denotational semantics — aims

- PCF types $\tau \;\mapsto\;$ domains $[\![\tau]\!]$.
- Closed PCF terms $M : \tau \;\mapsto\;$ elements $[\![M]\!] \in [\![\tau]\!]$.

  Denotations of open terms will be continuous functions.
- Compositionality.

  In particular: $[\![M]\!] = [\![M']\!] \;\Rightarrow\; [\![\mathcal{C}[M]]\!] = [\![\mathcal{C}[M']]\!]$.

PCF denotational semantics — aims

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- Soundness.

  For any type τ, $M \Downarrow_\tau V \;\Rightarrow\; [\![M]\!] = [\![V]\!]$.

PCF denotational semantics — aims

- PCF types $\tau \;\mapsto\;$ domains $[\![\tau]\!]$.

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- Soundness.

  For any type τ, $M \Downarrow_\tau V \;\Rightarrow\; [\![M]\!] = [\![V]\!]$.

- Adequacy.

  For $\tau = bool$ or nat, $[\![M]\!] = [\![V]\!] \in [\![\tau]\!] \;\Longrightarrow\; M \Downarrow_\tau V$.

Theorem. *For all types* τ *and closed terms* $M_1, M_2 \in \mathrm{PCF}_\tau$, *if* $[\![M_1]\!]$ *and* $[\![M_2]\!]$ *are equal elements of the domain* $[\![\tau]\!]$, *then* $M_1 \cong_{\mathrm{ctx}} M_2 : \tau$.

Assume $[\![M_1]\!] = [\![M_2]\!]$. Consider $\mathcal{C}[-] : \gamma$

$\mathcal{C}[M_1] \Downarrow V \implies [\![\mathcal{C}[M_1]]\!] = [\![V]\!]$ soundness

$\implies [\![\mathcal{C}[M_2]]\!] = [\![V]\!]$ compositionality

$\implies \mathcal{C}[M_2] \Downarrow V$ adequacy

$M_1 \lesssim_{ctx} M_2$

Theorem. *For all types τ and closed terms $M_1, M_2 \in \mathrm{PCF}_\tau$, if $[\![M_1]\!]$ and $[\![M_2]\!]$ are equal elements of the domain $[\![\tau]\!]$, then $M_1 \cong_{\mathrm{ctx}} M_2 : \tau$.*

Proof.

$$\mathcal{C}[M_1] \Downarrow_{nat} V \Rightarrow [\![\mathcal{C}[M_1]]\!] = [\![V]\!] \quad \text{(soundness)}$$

$$\Rightarrow [\![\mathcal{C}[M_2]]\!] = [\![V]\!] \quad \text{(compositionality on } [\![M_1]\!] = [\![M_2]\!]\text{)}$$

$$\Rightarrow \mathcal{C}[M_2] \Downarrow_{nat} V \quad \text{(adequacy)}$$

and symmetrically. $\square$

Proof principle

To prove

$$M_1 \cong_{\mathrm{ctx}} M_2 : \tau$$

it suffices to establish

$$[\![M_1]\!] = [\![M_2]\!] \text{ in } [\![\tau]\!]$$

$$\frac{[\![M_1]\!] = [\![M_2]\!]}{M_1 \cong_{ctx} M_2}$$

Proof principle

To prove

$$M_1 \cong_{\mathrm{ctx}} M_2 : \tau$$

it suffices to establish

$$[\![M_1]\!] = [\![M_2]\!] \text{ in } [\![\tau]\!]$$

? The proof principle is sound, but is it complete? That is, is equality in the denotational model also a necessary condition for contextual equivalence?

Topic 6

Denotational Semantics of PCF

Denotational semantics of PCF

To every typing judgement

$$\Gamma \vdash M : \tau$$

we associate a continuous function

$$[\![\Gamma \vdash M]\!] : [\![\Gamma]\!] \to [\![\tau]\!]$$

between domains.

Denotational semantics of PCF types

$$[\![nat]\!] \stackrel{\text{def}}{=} \mathbb{N}_\perp$$ (flat domain)

$$[\![bool]\!] \stackrel{\text{def}}{=} \mathbb{B}_\perp$$ (flat domain)

where $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $\mathbb{B} = \{true, false\}$.

Denotational semantics of PCF types

$$[\![nat]\!] \stackrel{\text{def}}{=} \mathbb{N}_\perp$$ (flat domain)

$$[\![bool]\!] \stackrel{\text{def}}{=} \mathbb{B}_\perp$$ (flat domain)

$$[\![\tau \to \tau']\!] \stackrel{\text{def}}{=} [\![\tau]\!] \to [\![\tau']\!]$$ (function domain).

where $\mathbb{N} = \{0, 1, 2, \dots\}$ and $\mathbb{B} = \{true, false\}$.

Ex: $[\![()]\!] = \{\bot\}$

Denotational semantics of PCF type environments

$$[\![\Gamma]\!] \stackrel{\text{def}}{=} \prod_{x \in dom(\Gamma)} [\![\Gamma(x)]\!]$$ (Γ-environments)

Γ is partial function from variables to types

If x is a variable in the domain of definition of Γ then $\Gamma(x)$ denotes the type associated to x in Γ.

$\Gamma \equiv (x_1 : \tau_1, \ldots, x_n : \tau_n) \sim \Gamma : [x_i \mapsto \tau_i]_{i=1 \ldots n}$

Ex:

$[\![x_1 : \tau_1, \ldots, x_n : \tau_n]\!] = [\![\tau_1]\!] \times \cdots \times [\![\tau_n]\!]$

$d \in = (d_1, \ldots, d_n)$ where $d_i \in [\![\tau_i]\!]$

Denotational semantics of PCF type environments

$$[\![\Gamma]\!] \stackrel{\text{def}}{=} \prod_{x \in dom(\Gamma)} [\![\Gamma(x)]\!] \qquad (\Gamma\text{-environments})$$

$=$ the domain of partial functions ρ from variables to domains such that $dom(\rho) = dom(\Gamma)$ and $\rho(x) \in [\![\Gamma(x)]\!]$ for all $x \in dom(\Gamma)$

Denotational semantics of PCF type environments

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Example:

1. For the empty type environment $\emptyset$,

$$[\![\emptyset]\!] = \{\,\bot\,\} = \emptyset_\bot$$

flat domain on the empty set

where $\bot$ denotes the unique partial function with $dom(\bot) = \emptyset$.

2. $[\![\langle x \mapsto \tau \rangle]\!] = \left(\{ x \} \to [\![\tau]\!]\right)$

2. $[\![\langle x \mapsto \tau \rangle]\!] = \big(\{ x \} \to [\![\tau]\!] \big) \cong [\![\tau]\!]$

2. $[\![\langle x \mapsto \tau \rangle]\!] \;=\; (\{\,x\,\} \to [\![\tau]\!]) \cong [\![\tau]\!]$

3.

$$
\begin{aligned}
& [\![\langle x_1 \mapsto \tau_1, \ldots, x_n \mapsto \tau_n \rangle]\!] \\
& \quad \cong \; (\{\,x_1\,\} \to [\![\tau_1]\!]) \times \ldots \times (\{\,x_n\,\} \to [\![\tau_n]\!]) \\
& \quad \cong \; [\![\tau_1]\!] \times \ldots \times [\![\tau_n]\!]
\end{aligned}
$$

$\Gamma \vdash M : \tau \;\rightsquigarrow\; [\![\Gamma \vdash M]\!] : [\![\Gamma]\!] \longrightarrow [\![\tau]\!]$ cont

Denotational semantics of PCF terms, I

$$[\![\Gamma \vdash \mathbf{0}]\!](\rho) \stackrel{\text{def}}{=} 0 \in [\![nat]\!]$$

$$[\![\Gamma \vdash \mathbf{true}]\!](\rho) \stackrel{\text{def}}{=} true \in [\![bool]\!]$$

$$[\![\Gamma \vdash \mathbf{false}]\!](\rho) \stackrel{\text{def}}{=} false \in [\![bool]\!]$$

$\rho \in [\![\Gamma]\!]$

$[\![\Gamma \vdash \underline{\underline{true}}]\!] = \lambda \rho \in [\![\Gamma]\!].\, true$

Denotational semantics of PCF terms, I

$$[\![\Gamma \vdash \mathbf{0}]\!](\rho) \stackrel{\text{def}}{=} 0 \in [\![nat]\!]$$

$$[\![\Gamma \vdash \mathbf{true}]\!](\rho) \stackrel{\text{def}}{=} true \in [\![bool]\!]$$

$$[\![\Gamma \vdash \mathbf{false}]\!](\rho) \stackrel{\text{def}}{=} false \in [\![bool]\!]$$

$$[\![\Gamma \vdash x]\!](\rho) \stackrel{\text{def}}{=} \rho(x) \in [\![\Gamma(x)]\!] \qquad (x \in dom(\Gamma))$$

$$[\![x_1 : \tau_1, \ldots, x_n : \tau_n \vdash x_i : \tau_i]\!]$$
$$= \lambda(\rho_1, \ldots, \rho_n) \in [\![\tau_1]\!] \times \cdots \times [\![\tau_n]\!].\ \rho_i$$

Denotational semantics of PCF terms, II

$$[\![\Gamma \vdash \mathbf{succ}(M)]\!](\rho)$$

$$\stackrel{\text{def}}{=} \begin{cases} [\![\Gamma \vdash M]\!](\rho) + 1 & \text{if } [\![\Gamma \vdash M]\!](\rho) \neq \bot \\ \bot & \text{if } [\![\Gamma \vdash M]\!](\rho) = \bot \end{cases}$$

Denotational semantics of PCF terms, II

$$\llbracket \Gamma \vdash \mathbf{succ}(M) \rrbracket(\rho)$$

$$\stackrel{\text{def}}{=} \begin{cases} \llbracket \Gamma \vdash M \rrbracket(\rho) + 1 & \text{if } \llbracket \Gamma \vdash M \rrbracket(\rho) \neq \bot \\ \bot & \text{if } \llbracket \Gamma \vdash M \rrbracket(\rho) = \bot \end{cases}$$

$$\llbracket \Gamma \vdash \mathbf{pred}(M) \rrbracket(\rho)$$

$$\stackrel{\text{def}}{=} \begin{cases} \llbracket \Gamma \vdash M \rrbracket(\rho) - 1 & \text{if } \llbracket \Gamma \vdash M \rrbracket(\rho) > 0 \\ \bot & \text{if } \llbracket \Gamma \vdash M \rrbracket(\rho) = 0, \bot \end{cases}$$

tied up to the operational semantics

Denotational semantics of PCF terms, II

$$[\![\Gamma \vdash \mathbf{succ}(M)]\!](\rho)$$

$$\stackrel{\text{def}}{=} \begin{cases} [\![\Gamma \vdash M]\!](\rho) + 1 & \text{if } [\![\Gamma \vdash M]\!](\rho) \neq \bot \\ \bot & \text{if } [\![\Gamma \vdash M]\!](\rho) = \bot \end{cases}$$

$$[\![\Gamma \vdash \mathbf{pred}(M)]\!](\rho)$$

$$\stackrel{\text{def}}{=} \begin{cases} [\![\Gamma \vdash M]\!](\rho) - 1 & \text{if } [\![\Gamma \vdash M]\!](\rho) > 0 \\ \bot & \text{if } [\![\Gamma \vdash M]\!](\rho) = 0, \bot \end{cases}$$

$$[\![\Gamma \vdash \mathbf{zero}(M)]\!](\rho) \stackrel{\text{def}}{=} \begin{cases} \mathit{true} & \text{if } [\![\Gamma \vdash M]\!](\rho) = 0 \\ \mathit{false} & \text{if } [\![\Gamma \vdash M]\!](\rho) > 0 \\ \bot & \text{if } [\![\Gamma \vdash M]\!](\rho) = \bot \end{cases}$$

Denotational semantics of PCF terms, III

$$[\![\Gamma \vdash \mathbf{if}\ M_1\ \mathbf{then}\ M_2\ \mathbf{else}\ M_3]\!](\rho)$$

$$\stackrel{\text{def}}{=} \begin{cases} [\![\Gamma \vdash M_2]\!](\rho) & \text{if } [\![\Gamma \vdash M_1]\!](\rho) = \mathit{true} \\ [\![\Gamma \vdash M_3]\!](\rho) & \text{if } [\![\Gamma \vdash M_1]\!](\rho) = \mathit{false} \\ \bot & \text{if } [\![\Gamma \vdash M_1]\!](\rho) = \bot \end{cases}$$

Denotational semantics of PCF terms, III

$$[\![\Gamma \vdash \textbf{if } M_1 \textbf{ then } M_2 \textbf{ else } M_3]\!](\rho)$$

$$\stackrel{\text{def}}{=} \begin{cases} [\![\Gamma \vdash M_2]\!](\rho) & \text{if } [\![\Gamma \vdash M_1]\!](\rho) = \mathit{true} \\ [\![\Gamma \vdash M_3]\!](\rho) & \text{if } [\![\Gamma \vdash M_1]\!](\rho) = \mathit{false} \\ \bot & \text{if } [\![\Gamma \vdash M_1]\!](\rho) = \bot \end{cases}$$

$$[\![\Gamma \vdash M_1\, M_2]\!](\rho) \stackrel{\text{def}}{=} \underbrace{\left([\![\Gamma \vdash M_1]\!](\rho)\right)}_{\in ([\![\sigma]\!] \to [\![\tau]\!])} \underbrace{\left([\![\Gamma \vdash M_2]\!](\rho)\right)}_{\in [\![\sigma]\!]}$$

M_1: $\sigma \to \tau$, M_2: σ

$$\frac{\Gamma, x : \tau \vdash M : \sigma}{\Gamma \vdash \mathbf{fn}\, x : \tau \,.\, M : \tau \to \sigma} \rightsquigarrow [\![M]\!] : \underbrace{[\![\Gamma, x : \tau]\!]}_{[\![\Gamma]\!] \times [\![\tau]\!]} \to [\![\sigma]\!]$$

Currying

$$[\![\mathbf{fn}\, x : \tau \,.\, M]\!] : [\![\Gamma]\!] \to ([\![\tau]\!] \to [\![\sigma]\!])$$

Denotational semantics of PCF terms, IV

$$[\![\Gamma \vdash \mathbf{fn}\, x : \tau \,.\, M]\!](\rho)$$

$$\stackrel{\text{def}}{=} \lambda d \in [\![\tau]\!] \,.\, [\![\Gamma[x \mapsto \tau] \vdash M]\!](\rho[x \mapsto d]) \qquad (x \notin dom(\Gamma))$$

$$\lambda d \in [\![\tau]\!] \,.\, [\![\Gamma, x : \tau \vdash M]\!](\rho, d)$$

NB: $\rho[x \mapsto d] \in [\![\Gamma[x \mapsto \tau]]\!]$ is the function mapping x to $d \in [\![\tau]\!]$ and otherwise acting like ρ.

Denotational semantics of PCF terms, V

$$[\![\Gamma \vdash \mathbf{fix}(M)]\!](\rho) \stackrel{\text{def}}{=} \mathit{fix}([\![\Gamma \vdash M]\!](\rho))$$

Recall that fix is the function assigning least fixed points to continuous functions.

Denotational semantics of PCF

Proposition. *For all typing judgements* $\Gamma \vdash M : \tau$, *the denotation*

$$[\![\Gamma \vdash M]\!] : [\![\Gamma]\!] \to [\![\tau]\!]$$

is a well-defined continous function.