Exercise Sheet 4 (graded, 25% of final course mark)

RETURN SOLUTIONS TO GRADUATE EDUCATION OFFICE BY 16:00 ON MONDAY 16 NOVEMBER

- $[\![c^A]\!] = [\![\Gamma]\!] \xrightarrow{\langle\rangle} 1 \xrightarrow{|c^A|} [\![A]\!]$

- $[\![x_i]\!] = [\![\Gamma]\!] \xrightarrow{\pi_i} [\![A_i]\!]$ if $\Gamma = [\dots, x_i : A_i, \dots]$

- $[\![()]\!] = [\![\Gamma]\!] \xrightarrow{\langle\rangle} 1 = [\![1]\!]$

- $[\![(s,t)]\!] = [\![\Gamma]\!] \xrightarrow{\langle [\![s]\!], [\![t]\!] \rangle} [\![A]\!] \times [\![B]\!] = [\![A \times B]\!]$

- $[\![\mathrm{fst}\ t]\!] = [\![\Gamma]\!] \xrightarrow{[\![t]\!]} [\![A \times B]\!] = [\![A]\!] \times [\![B]\!] \xrightarrow{\pi_1} [\![A]\!]$
  $[\![\mathrm{snd}\ t]\!] = \pi_2 \circ [\![t]\!]$

- $[\![\lambda x : A . t]\!] = \mathrm{cur}\left([\![\Gamma]\!] \times [\![A]\!] \cong [\![\Gamma, x : A]\!] \xrightarrow{[\![t]\!]} [\![B]\!] \right)$

- $[\![s\ t]\!] = [\![\Gamma]\!] \xrightarrow{\langle [\![s]\!], [\![t]\!] \rangle} [\![B]\!]^{[\![A]\!]} \times [\![A]\!] \xrightarrow{\mathrm{app}} [\![B]\!]$

Substitution

$t'[t/x]$ = result of replacing all free occurrences of variable x in term t' by the term t, α-converting λ-bound variables in t' to avoid them "capturing" any free variables of t

E.g. $(\lambda y : A . (y, x))[y/x]$ $\begin{cases} \text{is } \lambda z : A . (z, y) \\ \text{is } \underline{\text{NOT}}\ \lambda y : A . (y, y) \end{cases}$

Free variables $fv(t)$ of a term t

- $fv(c^A) = \emptyset$
- $fv(x) = \{x\}$
- $fv(()) = \emptyset$
- $fv((s,t)) = fv(s) \cup fv(t)$
- $fv(st) = fv(s) \cup fv(t)$
- $fv(\lambda x : A . t) = fv(t) - \{x\}$

$\boxed{\{ y \in fv(t) \mid y \neq x \}}$

Freshness relation:

$$x \# t \triangleq x \notin fv(t)$$

Substitution

$$\frac{}{c^A[t/x] = c^A}$$

$$\frac{}{x[t/x] = t}$$

$$\frac{y \neq x}{y[t/x] = y}$$

$$\frac{}{()[t/x] = ()}$$

$$\frac{s_1[t/x] = s_1' \qquad s_2[t/x] = s_2'}{(s_1, s_2)[t/x] = (s_1', s_2')}$$

$$\frac{s[t/x] = s'}{(\mathrm{fst}\ s)[t/x] = \mathrm{fst}\ s'}$$

$$\frac{s[t/x] = s'}{(\mathrm{snd}\ s)[t/x] = \mathrm{snd}\ s'}$$

$$\frac{s[t/x] = s' \qquad y \# (x, t)}{(\lambda y : A.\, s)[t/x] = \lambda y : A.\, s'}$$

$$\frac{s_1[t/x] = s_1' \qquad s_2[t/x] = s_2'}{(s_1\, s_2)[t/x] = s_1'\, s_2'}$$

Typing property of substitution

Substitution Lemma

If $\Gamma \vdash t : A$ & $\Gamma, x : A \vdash t' : A'$ (where $x \notin \Gamma$), then

$$\Gamma \vdash t'[t/x] : A'$$

Proof by induction on the structure of t'.

In case t' is $\lambda y : B . s$, need to first prove

Weakening Lemma If $\Gamma \vdash t : A$ & $x \notin \Gamma$, then

$$\Gamma, x : B \vdash t : A$$

$\beta\eta$-Equality $\Gamma \vdash s =_{\beta\eta} t : A$

where $\Gamma \vdash s : A$ & $\Gamma \vdash t : A$,

is inductively defined, as follows:

- β-Conversions

$$\frac{\Gamma, x : A \vdash t : B \qquad \Gamma \vdash s : A}{\Gamma \vdash (\lambda x : A . t)\, s =_{\beta\eta} t[s/x] : B}$$

$$\frac{\Gamma \vdash s : A \qquad \Gamma \vdash t : B}{\Gamma \vdash \mathrm{fst}(s,t) =_{\beta\eta} s : A} \qquad \frac{\Gamma \vdash s : A \qquad \Gamma \vdash t : B}{\Gamma \vdash \mathrm{snd}(s,t) =_{\beta\eta} t : B}$$

$\beta\eta$-Equality $\Gamma \vdash s =_{\beta\eta} t : A$

where $\Gamma \vdash s : A$ & $\Gamma \vdash t : A$,
is inductively defined, as follows:

- β-Conversions …
- η-Conversions

$$\frac{\Gamma \vdash t : A \to B \qquad x \# t}{\Gamma \vdash t =_{\beta\eta} (\lambda x : A .\, t\, x) : A \to B}$$

$$\frac{\Gamma \vdash t : A \times B}{\Gamma \vdash t =_{\beta\eta} (\mathrm{fst}\, t, \mathrm{snd}\, t) : A \times B} \qquad \frac{\Gamma \vdash t : 1}{\Gamma \vdash t =_{\beta\eta} () : 1}$$

$\beta\eta$-Equality $\Gamma \vdash s =_{\beta\eta} t : A$

where $\Gamma \vdash s : A$ & $\Gamma \vdash t : A$,

is inductively defined, as follows:

- β-Conversions...
- η-Conversions...
- Congruence rules

$$\frac{\Gamma, x : A \vdash t =_{\beta\eta} t' : B}{\Gamma \vdash \lambda x : A . t =_{\beta\eta} \lambda x : A . t' : A \to B}$$

$$\frac{\Gamma \vdash s =_{\beta\eta} s' : A \to B \qquad \Gamma \vdash t =_{\beta\eta} t' : A}{\Gamma \vdash s t =_{\beta\eta} s' t' : B}$$

etc.

$\beta\eta$-Equality $\Gamma \vdash s =_{\beta\eta} t : A$

where $\Gamma \vdash s : A$ & $\Gamma \vdash t : A$,
is inductively defined, as follows:

- β-Conversions...
- η-Conversions...
- Congruence rules...
- $=_{\beta\eta}$ is reflexive, symmetric & transitive

$$\frac{\Gamma \vdash t : A}{\Gamma \vdash t =_{\beta\eta} t : A}$$

etc.

$\beta\eta$-Equality $\Gamma \vdash s =_{\beta\eta} t : A$

Soundness Theorem for ccc semantics of STLC

If $\Gamma \vdash s =_{\beta\eta} t : A$, then in any ccc

$$[\![s]\!] = [\![t]\!] : [\![\Gamma]\!] \longrightarrow [\![A]\!]$$

Proof ... (uses following result about semantics of substitution...)

Semantics of substitution in a ccc

Theorem If $\Gamma \vdash t : A$ & $\Gamma, x : A \vdash t' : A'$
then in any ccc

$$\begin{tikzcd}
[\![\Gamma]\!] \arrow[r, "\langle id, [\![t]\!]\rangle"] \arrow[dr, "{[\![t'[t/x]]\!]}"'] & {[\![\Gamma]\!] \times [\![A]\!] \cong [\![\Gamma, x : A]\!]} \arrow[d, "{[\![t']\!]}"] \\
& {[\![A']\!]}
\end{tikzcd}$$

commutes

Proofs (by induction on structure of t & t') omitted

<u>Weakening lemma</u>
If $\Gamma \vdash t : A$ & $y \notin \Gamma$, then $[\![\Gamma, y : B]\!] \xrightarrow{[\![t]\!]} [\![A]\!]$
is equal to $[\![\Gamma, y : B]\!] \cong [\![\Gamma]\!] \times [\![B]\!] \xrightarrow{\pi_1} [\![\Gamma]\!] \xrightarrow{[\![t]\!]} [\![A]\!]$

The internal language of a ccc $\mathbb{C}$

- one ground type for each $\mathbb{C}$-object X
- one constant f^X for each $\mathbb{C}$-morphism $f : 1 \to X$ ("global element" of the object X)

Then types & terms of STLC over this language describe objects & morphisms of $\mathbb{C}$.

<u>For example</u> [Ex. Sh. 3, qu. 3], in any ccc $\mathbb{C}$ there is an isomorphism

$$\boxed{Z^{(X\times Y)} \cong (Z^Y)^X} \qquad (\text{any } X, Y, Z \in \mathrm{obj}\,\mathbb{C})$$

which in the internal language of $\mathbb{C}$ is described by terms

$$\boxed{\begin{array}{ll} s & \vdash \lambda f : (X\times Y)\to Z.\ \lambda x : X.\ \lambda y : Y.\ f(x,y) \\ & \quad : ((X\times Y)\to Z) \longrightarrow (X\to(Y\to Z)) \\ \hline t & \vdash \lambda g : (X\to(Y\to Z)).\ \lambda z : X\times Y.\ g(\mathrm{fst}\,z)(\mathrm{snd}\,z) \\ & \quad : (X\to(Y\to Z)) \to ((X\times Y)\to Z) \end{array}}$$

satisfying
$$\begin{cases} f : (X\times Y)\to Z \vdash t(s\,f) =_{\beta\eta} f \\ g : X\to(Y\to Z) \vdash s(t\,g) =_{\beta\eta} g \end{cases}$$