Theories as categories

Recall from earlier:

algebraic theory $\mathbb{T}$ $\mapsto$ Cartesian category $\mathbb{C}_{\mathbb{T}}$

The same construction applied to an equational theory in STLC yields a CCC.

(Exponential of $\Gamma = [x_1 : A_1, \ldots, x_n : A_n]$ & $\Delta = [y_1 : B_1, \ldots, y_m : B_m]$ in $\mathbb{C}_{\mathbb{T}}$ is $[f_1 : \vec{A} \to B_1, \ldots, f_m : \vec{A} \to B_m]$, where

$$\vec{A} \to B \triangleq A_1 \to (A_2 \to \cdots \to (A_n \to B) \cdots)$$

)

Theories as categories

Recall from earlier:

algebraic theory $\mathbb{T}$ $\longmapsto$ Cartesian category $\mathbb{C}_{\mathbb{T}}$

The same construction applied to an equational theory in STLC yields a CCC, but there's an equivalent & simpler construction...

Theories as categories

algebraic theory over STLC $\mathbb{T}$ $\longmapsto$ c.c.c $\mathcal{C}_{\mathbb{T}}$

- objects of $\mathcal{C}_{\mathbb{T}}$ are types of $\mathbb{T}$
- morphisms $A \to B$ in $\mathcal{C}_{\mathbb{T}}$ are equivalence classes of closed terms $\vdash t : A \to B$ for equiv. relation

$$\boxed{t \sim t' \text{ if } \vdash t = t' : A \to B \text{ is a } \mathbb{T} \text{ theorem}}$$

Curry-Howard-Lawvere

LOGIC		TYPE THEORY		CATEGORY TH.
propositions	$\leftrightarrow$	types	$\leftrightarrow$	objects
proofs	$\leftrightarrow$	terms	$\leftrightarrow$	morphisms

E.g. IPL vs STLC vs CCC's

proofs terms morphisms

Recall the derivation of $\varphi \Rightarrow \psi, \psi \Rightarrow \theta \vdash \varphi \Rightarrow \theta$ in IPL:

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{(Ax)}
\UnaryInfC{$\Phi \vdash \psi \Rightarrow \theta$}
\AxiomC{}
\RightLabel{(Ax)}
\UnaryInfC{$\Phi \vdash \varphi \Rightarrow \psi$}
\AxiomC{}
\RightLabel{(Ax)}
\UnaryInfC{$\Phi \vdash \varphi$}
\RightLabel{($\Rightarrow$E)}
\BinaryInfC{$\Phi \vdash \psi$}
\RightLabel{($\Rightarrow$E)}
\BinaryInfC{$\Phi \vdash \theta$}
\RightLabel{($\Rightarrow$I)}
\UnaryInfC{$\varphi \Rightarrow \psi, \psi \Rightarrow \theta \vdash \varphi \Rightarrow \theta$}
\end{prooftree}
$$

$$\left(\Phi \triangleq \varphi \Rightarrow \psi, \psi \Rightarrow \theta, \varphi \right)$$

A corresponding STLC term:

$$\dfrac{\dfrac{\dfrac{}{\Phi \vdash z : \psi \Rightarrow \theta}\,(\mathrm{var}) \qquad \dfrac{\dfrac{}{\Phi \vdash y : \varphi \Rightarrow \psi}\,(\mathrm{var}) \quad \dfrac{}{\Phi \vdash x : \varphi}\,(\mathrm{var})}{\Phi \vdash yx : \psi}\,(\mathrm{app})}{\Phi \vdash z(yx) : \theta}\,(\mathrm{app})}{y : \varphi \Rightarrow \psi,\ z : \psi \Rightarrow \theta \vdash \lambda x{:}\varphi.\ z(yx) : \varphi \Rightarrow \theta}\,(\lambda)$$

$$\left(\Phi \triangleq [\,y : \varphi \Rightarrow \psi,\ z : \psi \Rightarrow \theta,\ x : \varphi\,]\right)$$

Curry-Howard-Lawvere

these correspondences can be made into categorical <u>equivalences</u> - need the notions of <u>functor</u> & <u>natural transformation</u> to define "equivalence"...

E.g. IPL vs STLC vs CCC's

cut-free proofs | terms | morphisms

Functors

are the appropriate notion of morphism between categories. Given categories $\mathbb{C}$ & $\mathbb{D}$, a **functor** $\boxed{F : \mathbb{C} \to \mathbb{D}}$ is specified by:

- a function $\mathrm{Obj}\,\mathbb{C} \longrightarrow \mathrm{Obj}\,\mathbb{D}$ whose value at a $\mathbb{C}$-object X is written FX
- for all $\mathbb{C}$-objects X & Y, a function $\mathbb{C}(X,Y) \longrightarrow \mathbb{D}(FX,FY)$ whose value at a $\mathbb{C}$-morphism $f : X \to Y$ is written $Ff : FX \to FY$

satisfying
$$\begin{cases} F(g \circ f) = Fg \circ Ff \\ F(id_X) = id_{FX} \end{cases}$$

Examples of functors

"forgetful" functors from categories of sets-with-structure back to **Set**

E.g. $\boxed{U : \mathbf{Mon} \to \mathbf{Set}}$

$$\begin{cases} U(M, \cdot, e) \triangleq M \\ U\big((M, \cdot, e) \xrightarrow{f} (N, \cdot, e)\big) \triangleq M \xrightarrow{f} N \end{cases}$$

and similarly $U : \mathbf{Pre} \to \mathbf{Set}$

Examples of functors

Free monoid functor $\boxed{F : \mathbf{Set} \to \mathbf{Mon}}$

Recall free monoid on a set Σ is

$$(\underbrace{List(\Sigma)}_{\text{finite lists of elements of } \Sigma}, \underbrace{@}_{\text{list concatenation}}, \underbrace{nil}_{\text{empty list}})$$

Examples of functors

Free monoid functor $\boxed{F : \mathbf{Set} \to \mathbf{Mon}}$

Recall free monoid on a set Σ is

$$F(\Sigma) \triangleq (\mathrm{List}(\Sigma), @, nil)$$

Given $f \in \mathbf{Set}(\Sigma_1, \Sigma_2)$ we get

$F(f) : F(\Sigma_1) \to F(\Sigma_2)$ mapping each list

$l = [a_1, \dots, a_n] \in \Sigma_1^*$ to $Ffl \triangleq [fa_1, \dots, fa_n]$

Easy to see that $F(id_\Sigma) = id_{F(\Sigma)}$ &

$$F(g \circ f) = (Fg) \circ (Ff)$$

Examples of functors

If $\mathbb{C}$ is a cartesian category and $X \in \text{obj}\,\mathbb{C}$, then

$$Y \in \text{obj}\,\mathbb{C} \mapsto Y \times X$$

extends to a functor $\boxed{(\_) \times X : \mathbb{C} \to \mathbb{C}}$

via $(Y \xrightarrow{f} Y') \mapsto (Y \times X \xrightarrow{f \times id_X} Y' \times X)$

since $\begin{cases} id_Y \times id_X = id_{Y \times X} \\ (g \circ f) \times id_X = (g \times id_X) \circ (f \times id_X) \end{cases}$

[see Ex. Sh. 2, q1c]

Examples of functors

If $\mathbb{C}$ is a cartesian closed category and $X \in \mathrm{obj}\,\mathbb{C}$, then

$$Y \in \mathrm{obj}\,\mathbb{C} \mapsto Y^X$$

extends to a functor $\boxed{(\_)^X : \mathbb{C} \to \mathbb{C}}$

via

$$(Y \xrightarrow{f} Y') \mapsto (Y^X \xrightarrow{f^X} Y'^X)$$

$$f^X = \mathrm{cur}(f \circ \mathrm{app})$$

Since $\begin{cases} \mathrm{id}^X = \mathrm{id} \\ (g \circ f)^X = g^X \circ f^X \end{cases}$

[see Ex. Sh. 3, q 7]

Contravariance

A functor $F : \mathbb{C}^{op} \to \mathbb{D}$ is called a **contravariant functor from $\mathbb{C}$ to $\mathbb{D}$**

Note that if $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathbb{C}$

then $X \xleftarrow{f} Y \xleftarrow{g} Z$ in $\mathbb{C}^{op}$

so

$$FX \xleftarrow{Ff} FY \xleftarrow{Fg} FZ \quad \text{in } \mathbb{D}$$

$$F(g \circ f) = Ff \circ Fg$$

composition in $\mathbb{C}$

composition in $\mathbb{D}$

Example of contravariant functor

If $\mathbb{C}$ is a cartesian closed category and $X \in \mathrm{obj}\,\mathbb{C}$, then

$$Y \in \mathrm{obj}\,\mathbb{C} \mapsto X^Y$$

extends to a functor $\boxed{X^{(-)} : \mathbb{C}^{op} \to \mathbb{C}}$

via $\left(Y \xrightarrow{f} Y'\right) \mapsto \left(X^Y \xleftarrow{X^f} X^{Y'}\right)$

$$X^f = cur(app \circ (id \times f))$$

Since $\begin{cases} X^{id} = id \\ X^{g \circ f} = X^f \circ X^g \end{cases}$

[see Ex. Sh. 3, q8]

Note that since a functor $F: \mathbb{C} \to \mathbb{D}$ preserves domains, codomains, composition

it sends commutative diagrams in $\mathbb{C}$ to commutative diagrams in $\mathbb{D}$

E.g.

$$
\begin{tikzcd}
X \arrow[r, bend left, "f"] \arrow[d, bend right, "h"'] & Y \arrow[dl, bend left, "g"] \\
Z &
\end{tikzcd}
\quad \overset{F}{\longmapsto} \quad
\begin{tikzcd}
FX \arrow[r, bend left, "Ff"] \arrow[d, bend right, "Fh"'] & FY \arrow[dl, bend left, "Fg"] \\
FZ &
\end{tikzcd}
$$

$$Fh = F(g \circ f) = Fg \circ Ff$$

Note ~~that~~ since a functor $F : \mathbb{C} \to \mathbb{D}$ preserves domains, codomains, composition & identities,

it sends isomorphisms in $\mathbb{C}$ to isos in $\mathbb{D}$ because

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[dr, "id_X"'] & Y \arrow[d, "g"] \arrow[dr, "id_Y"] & \\
& X \arrow[r, "f"'] & Y
\end{tikzcd}
\quad \overset{F}{\longmapsto} \quad
\begin{tikzcd}
FX \arrow[r, "Ff"] \arrow[dr, "id_{FX}"'] & FY \arrow[d, "Fg"] \arrow[dr, "id_{FY}"] & \\
& FX \arrow[r, "Ff"'] & FY
\end{tikzcd}
$$

So $\boxed{F(f^{-1}) = (Ff)^{-1}}$

Composing functors

Given functors

$$\mathbb{C} \xrightarrow{F} \mathbb{D} \xrightarrow{G} \mathbb{E}$$

we get a functor $\boxed{G \circ F : \mathbb{C} \to \mathbb{E}}$

$$G \circ F \begin{pmatrix} X \\ \downarrow f \\ Y \end{pmatrix} \triangleq \begin{array}{c} G(FX) \\ \downarrow G(Ff) \\ G(FY) \end{array}$$

(this preserves composition & identities because F & G do so)

Identity functor

on a category $\mathbb{C}$ is

$$\boxed{Id_{\mathbb{C}} : \mathbb{C} \longrightarrow \mathbb{C}}$$

where

$$Id_{\mathbb{C}}\begin{pmatrix} X \\ \downarrow f \\ Y \end{pmatrix} \triangleq \begin{matrix} X \\ \downarrow f \\ Y \end{matrix}$$

Functor composition satisfies the usual category laws

associativity $\quad H\circ(G\circ F) = (H\circ G)\circ F$

unity $\quad Id_{\mathbb{D}}\circ F = F = F\circ Id_{\mathbb{C}}$

So we can get categories whose
objects are categories
morphisms are functors
but we have to be a bit carefull about size...