Functors

are the appropriate notion of morphism between categories. Given categories $\mathbb{C}$ & $\mathbb{D}$, a functor $\boxed{F : \mathbb{C} \to \mathbb{D}}$ is specified by:

- a function $\mathrm{Obj}\,\mathbb{C} \longrightarrow \mathrm{Obj}\,\mathbb{D}$ whose value at a $\mathbb{C}$-object X is written FX
- for all $\mathbb{C}$-objects X & Y, a function $\mathbb{C}(X,Y) \longrightarrow \mathbb{D}(FX,FY)$ whose value at a $\mathbb{C}$-morphism $f : X \to Y$ is written $Ff : FX \to FY$

satisfying
$$\begin{cases} F(g \circ f) = Fg \circ Ff \\ F(id_X) = id_{FX} \end{cases}$$

Composing functors

Given functors

$$\mathbb{C} \xrightarrow{F} \mathbb{D} \xrightarrow{G} \mathbb{E}$$

we get a functor $\boxed{G \circ F : \mathbb{C} \to \mathbb{E}}$

$$G \circ F \left(\begin{array}{c} X \\ \downarrow f \\ Y \end{array} \right) \triangleq \begin{array}{c} G(FX) \\ \downarrow G(Ff) \\ G(FY) \end{array}$$

(this preserves composition & identities because F & G do so)

Identity functor

on a category $\mathbb{C}$ is

$$\boxed{\mathrm{Id}_{\mathbb{C}} : \mathbb{C} \longrightarrow \mathbb{C}}$$

where

$$\mathrm{Id}_{\mathbb{C}}\left(\begin{array}{c} X \\ \downarrow f \\ Y \end{array}\right) \triangleq \begin{array}{c} X \\ \downarrow f \\ Y \end{array}$$

Functor composition satisfies the usual category laws

associativity $H\circ(G\circ F) = (H\circ G)\circ F$

unity $Id_{\mathbb{D}}\circ F = F = F\circ Id_{\mathbb{C}}$

So we can get categories whose
objects are categories
morphisms are functors

but we have to be a bit carefull about size...

Russell's Paradox

Urestricted use of set comprehension

$\{x \mid \varphi(x)\}$ the set of all objects x that have property $\varphi(x)$

leads to contradiction (a proof of false),

since $R \triangleq \{x \mid x \notin x\}$

satisfies $R \in R \iff R \notin R$,

which is logically equivalent to false.

Size

We can't form the "set of <u>all</u> sets"
or "the category of <u>all</u> categories"

because we assume

$\boxed{\text{set membership is a well-founded relation — there can be no infinite sequence of sets } X_0, X_1, X_2, \ldots \text{ with } \cdots X_{n+1} \in X_n \in \cdots \in X_2 \in X_1 \in X_0}$

so in particular, there is no set X with $X \in X$

Size

We can't form the "set of <u>all</u> sets"
or "the category of <u>all</u> categories"

but we do assume there are (lots of)
big sets

$$\mathcal{U}_0 \in \mathcal{U}_1 \in \mathcal{U}_2 \in \cdots$$

where each $\mathcal{U}_i$ is a Grothendieck
universe ...

A Grothendieck Universe, $\mathcal{U}$

is a set of sets satisfying

- $x \in y \in \mathcal{U} \Rightarrow x \in \mathcal{U}$
- $x, y \in \mathcal{U} \Rightarrow \{x, y\} \in \mathcal{U}$
- $x \in \mathcal{U} \Rightarrow \mathcal{P}x \triangleq \{y \mid y \subseteq x\} \in \mathcal{U}$
- $X \in \mathcal{U}$ & $F \in \mathcal{U}^X \Rightarrow \{y \mid (\exists x \in X)\, y \in Fx\} \in \mathcal{U}$

and hence also

$$x, y \in \mathcal{U} \Rightarrow x \times y \in \mathcal{U} \;\&\; y^x \in \mathcal{U}.$$

The above properties are satisfied by $\mathcal{U} = \emptyset$, but we will always assume

- (axiom of infinity) $\mathbb{N} \in \mathcal{U}$

Size

We assume there is an infinite sequence
$$\mathcal{U}_0 \in \mathcal{U}_1 \in \mathcal{U}_2 \in \cdots$$
of bigger & bigger Grothendieck universes.

and revise our previous definition of "the" category of sets:

$\mathbf{Set}_i \triangleq$ category whose objects are all the elements of $\mathcal{U}_i$ and with $\mathbf{Set}_i(X, Y) = Y^X =$ all functions from X to Y

$\mathbf{Set} \triangleq \mathbf{Set}_0$ – its objects are called small sets (and other sets we call large)

Size

$\mathbf{Set}$ is the category of small sets.

<u>Definition</u> A category $\mathbb{C}$ is locally small if for all $X, Y \in \mathrm{obj}\,\mathbb{C}$, $\mathbb{C}(X,Y) \in \mathbf{Set}$

$\mathbb{C}$ is a small category if it is both locally small & $\mathrm{obj}\,\mathbb{C} \in \mathbf{Set}$

E.g. $\mathbf{Set}$, $\mathbf{Pre}$, $\mathbf{Mon}$ are all locally small (but not small).
Each $P \in \mathbf{Pre}$ & $M \in \mathbf{Mon}$ determines a small category.

The category of small categories, $\mathbf{Cat}$

- objects are all small categories
- morphisms $\mathbf{Cat}(\mathbb{C}, \mathbb{D})$ are all functors $F : \mathbb{C} \to \mathbb{D}$
- Composition & identities – for functors, as before

Cat is cartesian

- Terminal object in Cat is

$$\boxed{\begin{tikzcd} * \arrow[loop right, "\mathrm{id}_*"] \end{tikzcd}} = \text{one-object, one morphism category}$$

Cat is cartesian

- Binary product $\mathbb{C} \xleftarrow{\pi_1} \mathbb{C}\times\mathbb{D} \xrightarrow{\pi_2} \mathbb{D}$
  - objects of $\mathbb{C}\times\mathbb{D}$ are pairs (X, Y) with $X \in \mathrm{Obj}\,\mathbb{C}$ & $Y \in \mathrm{Obj}\,\mathbb{D}$
  - morphisms $(X,Y) \to (X',Y')$ are pairs (f,g) of morphisms $f \in \mathbb{C}(X,X')$, $g \in \mathbb{D}(Y,Y')$
  - composition & identities as in $\mathbb{C}$ & $\mathbb{D}$
  - $\pi_1(X,Y) = X \qquad \pi_1(f,g) = f$
    $\pi_2(X,Y) = Y \qquad \pi_2(f,g) = g$

Cat is not only cartesian, it is also cartesian closed – exponentials in Cat are called functor categories and to define them we need to consider natural transformations which are the appropriate notion of morphism between functors.