

Adjoint functors

Categories, functors & natural transformations were invented (by Eilenberg & MacLane) in order to formalize "adjoint situations".

They appear everywhere in mathematics, logic and (hence) CS.

Examples that we have seen ...

Binary product in $\mathcal{C}$

morphisms
in $\mathcal{C} \times \mathcal{C}$

$$(Z, Z) \rightarrow (X, Y)$$

$$\underline{\underline{Z \rightarrow X \times Y}}$$

morphisms in $\mathcal{C}$

bijective correspondence :

$$\mathcal{C} \times \mathcal{C} \left((Z, Z), (X, Y) \right) \cong \mathcal{C} (Z, X \times Y)$$

$$(f, g) \longmapsto \langle f, g \rangle$$

$$(\pi_1 \circ h, \pi_2 \circ h) \longleftarrow h$$

furthermore, this bijection "is natural in X, Y, Z "
(to be explained)

Exponentials in $\mathcal{C}$

$$\underline{Z \times X \longrightarrow Y}$$

morphisms in $\mathcal{C}$

$$Z \longrightarrow Y^X$$

morphisms in $\mathcal{C}$

bijective correspondence

$$\mathcal{C}(Z \times X, Y) \cong \mathcal{C}(Z, Y^X)$$

$$f \longmapsto \text{cur } f$$

$$\text{app} \circ (g \times \text{id}_X) \longleftarrow g$$

natural in X, Y, Z

Free monoids

$$\underline{\underline{\Sigma \rightarrow \mathcal{U}(M, \cdot, 1_M)}} \quad \text{in Set}$$

$$F\Sigma \rightarrow (M, \cdot, 1_M) \quad \text{in Mon}$$

↑ free monoid on set Σ
 $(\text{List}(\Sigma), _ @ _, \text{nil})$

bijection correspondence

$$\text{Set}(\Sigma, \mathcal{U}M) \cong \text{Mon}_{\wedge}(F\Sigma, M)$$

$$\begin{array}{ccc} f & \xrightarrow{\quad} & f \\ g \circ i_\Sigma & \xleftarrow{\quad} & g \end{array}$$

natural in Σ & M

Adjunction

Definition An **adjunction** between two categories $\mathcal{C}$ & $\mathcal{D}$ is specified by

- functors $\mathcal{C} \begin{matrix} \xrightarrow{F} \\ \xleftarrow{G} \end{matrix} \mathcal{D}$

- bijections

$$\theta_{x,y} : \mathcal{D}(F(x), y) \cong \mathcal{C}(x, G(y))$$

for each $x \in \text{Obj } \mathcal{C}$ & $y \in \text{Obj } \mathcal{D}$

which are **natural in x & y** , meaning...

For $\theta_{x,y}: \mathbb{D}(F(x), Y) \cong \mathbb{C}(X, G(Y))$

to be "natural in X & Y " means

for all $\begin{cases} u: X' \rightarrow X \text{ in } \mathbb{C} \\ v: Y \rightarrow Y' \text{ in } \mathbb{D} \end{cases}$

and all $g: F(X) \rightarrow Y$ in $\mathbb{D}$

$$X' \xrightarrow{u} X \xrightarrow{\theta_{x,y}(g)} G(Y) \xrightarrow{Gv} G(Y')$$

$$= \theta_{x',y'} (F(X') \xrightarrow{Fu} F(X) \xrightarrow{g} Y \xrightarrow{v} Y')$$

for $\theta_{x,y}: \mathbb{D}(F(x), Y) \cong \mathbb{C}(X, G(Y))$

to be "natural in X & Y " means

for all $\begin{cases} u: X' \rightarrow X \text{ in } \mathbb{C} \\ v: Y \rightarrow Y' \text{ in } \mathbb{D} \end{cases}$

and all $g: F(X) \rightarrow Y$ in $\mathbb{D}$

$$\begin{aligned} X' &\xrightarrow{u} X \xrightarrow{\theta_{x,y}(g)} G(Y) \xrightarrow{Gv} G(Y') \\ &= \theta_{x',y'} \quad F(X') \xrightarrow{Fu} F(X) \xrightarrow{g} Y \xrightarrow{v} Y' \end{aligned}$$

what has this
to do with
the concept of
natural
transformation?

Hom functors

If $\mathcal{C}$ is locally small, then we get a functor

$$H_{\mathcal{C}}: \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \text{Set}$$

with $H_{\mathcal{C}}(X, Y) \triangleq \mathcal{C}(X, Y)$ and

$$H_{\mathcal{C}}\left((X, Y) \xrightarrow{(f, g)} (X', Y')\right) \triangleq \mathcal{C}(X, Y) \rightarrow \mathcal{C}(X', Y')$$

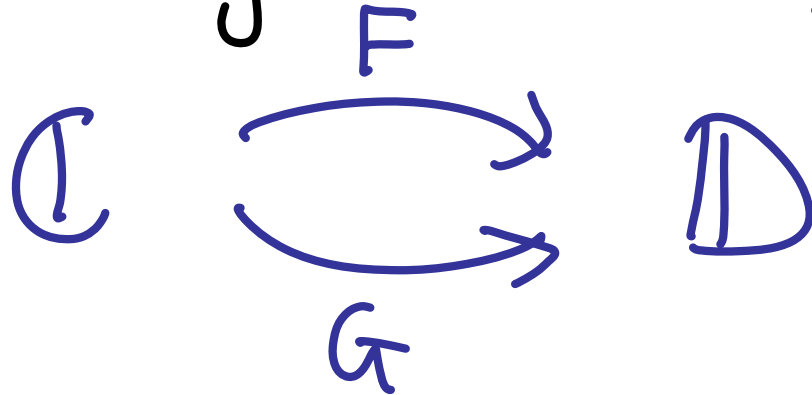
$h \mapsto g \circ h \circ f$

$f: X' \rightarrow X$
in $\mathcal{C}$

$g: Y \rightarrow Y'$
in $\mathcal{C}$

Natural isomorphisms

Given categories and functors

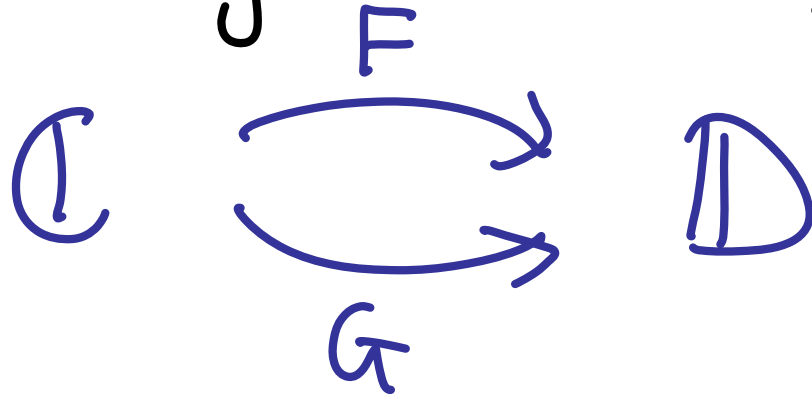


a **natural isomorphism** $\theta: F \cong G$

is simply an isomorphism between F & G in the functor category $\mathcal{D}^{\mathcal{C}}$.

Natural isomorphisms

Given categories and functors



FACT If $\theta : F \rightarrow G$ is a nat. transf.ⁿ
and for each $x \in \text{Obj } \mathbb{C}$, $\theta_x : F(x) \rightarrow G(x)$
is an isomorphism in $\mathbb{D}$, then
 $\theta_x^{-1} : G(x) \rightarrow F(x)$ gives a nat. transf.ⁿ
 $\theta^{-1} : G \rightarrow F$ & $F \cong G$ in $\mathbb{D}^{\mathbb{C}}$

Given locally small categories $\mathcal{C}$ & $\mathcal{D}$,
 if we have $\mathcal{C} \begin{matrix} \xrightarrow{F} \\ \xleftarrow{G} \end{matrix} \mathcal{D}$ we get
 functors

$$\begin{array}{ccccc}
 & F^{\text{op}} \times \text{id} & \rightarrow & \mathcal{D}^{\text{op}} \times \mathcal{D} & \xrightarrow{H_{\mathcal{D}}} \\
 \mathcal{C}^{\text{op}} \times \mathcal{D} & & & & \searrow \\
 & \text{id} \times G & \rightarrow & \mathcal{C}^{\text{op}} \times \mathcal{C} & \xrightarrow{H_{\mathcal{C}}} \text{Set}
 \end{array}$$

An adjunction (F, G, θ) is give by a
 nat. iso $\theta : H_{\mathcal{D}} \circ (F^{\text{op}} \times \text{id}) \cong H_{\mathcal{C}} \circ (\text{id} \times G)$

Terminology Given $\mathbb{C} \begin{matrix} \xrightarrow{F} \\ \xleftarrow{G} \end{matrix} \mathbb{D}$,

if there is some $\theta : H_{\mathbb{D}}^{\circ}(F^{\text{op}} \times \text{id}) \cong H_{\mathbb{C}}^{\circ}(\text{id} \times G)$
one says

F is a left adjoint for G

G is a right adjoint for F

and writes

$$F \dashv G$$

Notation associated with an adjunction
 (F, G, θ)

Given $\begin{cases} g: FX \rightarrow Y \\ f: X \rightarrow GX \end{cases}$

we write $\begin{cases} \bar{g} \triangleq \theta_{X,Y}(g) : X \rightarrow GX \\ \bar{f} \triangleq \theta_{X,Y}^{-1}(f) : FX \rightarrow Y \end{cases}$

Thus $\bar{g} = g$, $\bar{f} = f$ and naturality means

$$\overline{v \circ g \circ fu} = Gv \circ \bar{g} \circ u$$

The existence of θ is sometimes indicated by writing

$$\frac{Fx \xrightarrow{g} Y}{X \xrightarrow{\bar{g}} GX}$$

The existence of θ is sometimes indicated by writing

$$\theta \curvearrowright \frac{Fx \xrightarrow{g} Y}{X \xrightarrow{\bar{g}} GY} \curvearrowleft \theta^{-1}$$

Using this notation, can split the naturality condition for θ into two :

$$\frac{Fx' \xrightarrow{Fu} Fx \xrightarrow{g} Y}{X' \xrightarrow{u} X \xrightarrow{\bar{g}} GY}$$

$$\frac{Fx \xrightarrow{g} Y \xrightarrow{v} Y'}{X \xrightarrow{\bar{g}} GY \xrightarrow{Gv} GY'}$$

Proposition. $\mathcal{C}$ has binary products if & only if the diagonal functor $\Delta = \langle \text{id}, \text{id} \rangle : \mathcal{C} \rightarrow \mathcal{C} \times \mathcal{C}$ has a right adjoint.

Proposition A cartesian category $\mathcal{C}$ has all exponentials if & only if for all $X \in \text{Obj } \mathcal{C}$, the functor $(-) \times X : \mathcal{C} \rightarrow \mathcal{C}$ has a right adjoint.