Adjoint functors

Categories, functors & natural transformations were invented (by Eilenberg & MacLane) in order to formalize "adjoint situations"

They appear everywhere in mathematics, logic and (hence) CS.

Examples that we have seen ...

Binary product in $\mathbb{C}$

morphisms in $\mathbb{C} \times \mathbb{C}$

$$\frac{(Z, Z) \to (X, Y)}{Z \to X \times Y}$$

morphisms in $\mathbb{C}$

bijective correspondence:

$$\mathbb{C} \times \mathbb{C}\big((Z,Z),(X,Y)\big) \cong \mathbb{C}(Z, X \times Y)$$

$$(f, g) \longmapsto \langle f, g \rangle$$

$$(\pi_1 \circ h, \pi_2 \circ h) \longleftarrow\!\shortmid\; h$$

Furthermore, this bijection "is natural in X, Y, Z"
(to be explained)

Exponentials in $\mathbb{C}$

$$\frac{Z \times X \longrightarrow Y}{\overline{Z \longrightarrow Y^X}}$$

morphisms in $\mathbb{C}$

morphisms in $\mathbb{C}$

bijective correspondence

$$\mathbb{C}(Z \times X, Y) \cong \mathbb{C}(Z, Y^X)$$

$$f \longmapsto \mathrm{cur}\, f$$

$$\mathrm{app} \circ (g \times \mathrm{id}_X) \longleftarrow\!\shortmid\; g$$

natural in X, Y, Z

Free monoids

$$\frac{\Sigma \longrightarrow U(M, \cdot, 1_m) \quad \text{in } Set}{F\Sigma \longrightarrow (M, \cdot, 1_m) \quad \text{in } Mon}$$

free monoid on set Σ
$(List(\Sigma), \_@\_, nil)$

bijective correspondence

$$Set(\Sigma, UM) \cong Mon(F\Sigma, M)$$
$$f \longmapsto \hat{f}$$
$$g \circ i_\Sigma \longleftarrow g$$

natural in Σ & M

Adjunction

<u>Definition</u> An adjunction between two categories $\mathbb{C}$ & $\mathbb{D}$ is specified by

- functors $\begin{tikzcd} \mathbb{C} \arrow[r, bend left, "F"] & \mathbb{D} \arrow[l, bend left, "G"] \end{tikzcd}$

- bijections

$$\theta_{X,Y} : \mathbb{D}(F(X), Y) \cong \mathbb{C}(X, G(Y))$$

for each $X \in \mathrm{obj}\,\mathbb{C}$ & $Y \in \mathrm{obj}\,\mathbb{D}$ which are natural in X & Y, meaning...

For $\theta_{X,Y} : \mathbb{D}(F(X), Y) \cong \mathbb{C}(X, G(Y))$

to be "natural in X & Y" means

for all $\begin{cases} u : X' \to X \text{ in } \mathbb{C} \\ v : Y \to Y' \text{ in } \mathbb{D} \end{cases}$

and all $g : F(X) \to Y$ in $\mathbb{D}$

$$X' \xrightarrow{u} X \xrightarrow{\theta_{X,Y}(g)} G(Y) \xrightarrow{Gv} G(Y')$$

$$= \theta_{X',Y'}\left(F(X') \xrightarrow{Fu} F(X) \xrightarrow{g} Y \xrightarrow{v} Y'\right)$$

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what has this to do with the concept of natural transformation?

Hom functors

If $\mathbb{C}$ is locally small, then we get a functor

$$\boxed{H_{\mathbb{C}} : \mathbb{C}^{op} \times \mathbb{C} \longrightarrow \mathbf{Set}}$$

with $H_{\mathbb{C}}(X,Y) \triangleq \mathbb{C}(X,Y)$ and

$$H_{\mathbb{C}}\left((X,Y) \xrightarrow{(f,g)} (X',Y')\right) \triangleq \begin{array}{ccc} \mathbb{C}(X,Y) & \longrightarrow & \mathbb{C}(X',Y') \\ h & \longmapsto & g \circ h \circ f \end{array}$$

$f : X' \to X$ in $\mathbb{C}$

$g : Y \to Y'$ in $\mathbb{C}$

Natural isomorphisms

Given categories and functors

$$\begin{tikzcd} \mathbb{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] & \mathbb{D} \end{tikzcd}$$

a natural isomorphism $\theta : F \cong G$ is simply an isomorphism between F & G in the functor category $\mathbb{D}^{\mathbb{C}}$.

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Given categories and functors

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<u>FACT</u> If $\theta : F \to G$ is a nat. transf.^{n} and for each $X \in \mathrm{obj}\,\mathbb{C}$, $\theta_X : F(X) \to G(X)$ is an isomorphism in $\mathbb{D}$, then $\theta_X^{-1} : G(X) \to F(X)$ gives a nat. transf^{n} $\theta^{-1} : G \to F$ & $F \cong G$ in $\mathbb{D}^{\mathbb{C}}$.

Given locally small categories $\mathbb{C}$ & $\mathbb{D}$, if we have $\mathbb{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathbb{D}$ we get functors

$$\begin{tikzcd}
 & \mathbb{D}^{op} \times \mathbb{D} \arrow[dr, "H_{\mathbb{D}}"] & \\
\mathbb{C}^{op} \times \mathbb{D} \arrow[ur, "F^{op} \times id"] \arrow[dr, "id \times G"'] & & \mathbf{Set} \\
 & \mathbb{C}^{op} \times \mathbb{C} \arrow[ur, "H_{\mathbb{C}}"'] &
\end{tikzcd}$$

An adjunction (F, G, θ) is give by a nat. iso $\theta : H_{\mathbb{D}} \circ (F^{op} \times id) \cong H_{\mathbb{C}} \circ (id \times G)$

Terminology

Given
$$\begin{tikzcd} \mathbb{C} \arrow[r, bend left, "F"] & \mathbb{D} \arrow[l, bend left, "G"] \end{tikzcd},$$

if there is some $\theta : H_{\mathbb{D}} \circ (F^{op} \times id) \cong H_{\mathbb{C}} \circ (id \times G)$

one says

F is a left adjoint for G

G is a right adjoint for F

and writes $\boxed{F \dashv G}$

Notation associated with an adjunction (F, G, θ)

Given $\begin{cases} g: FX \to Y \\ f: X \to GY \end{cases}$

we write $\begin{cases} \bar{g} \triangleq \theta_{X,Y}(g) : X \to GY \\ \bar{f} \triangleq \theta_{X,Y}^{-1}(f) : FX \to Y \end{cases}$

Thus $\bar{\bar{g}} = g$, $\bar{\bar{f}} = f$ and naturality means

$$\frac{}{v \circ g \circ Fu} = Gv \circ \bar{g} \circ u$$

The existence of θ is sometimes indicated by writing

$$\boxed{\dfrac{Fx \xrightarrow{g} Y}{X \xrightarrow{\bar{g}} GY}}$$

The existence of θ is sometimes indicated by writing

$$\theta \downarrow \; \boxed{\dfrac{FX \xrightarrow{g} Y}{X \xrightarrow{\bar{g}} GY}} \; \uparrow \theta^{-1}$$

Using this notation, can split the naturality condition for θ into two :

$$\dfrac{FX' \xrightarrow{Fu} FX \xrightarrow{g} Y}{X' \xrightarrow{u} X \xrightarrow{\bar{g}} GY} \qquad \dfrac{FX \xrightarrow{g} Y \xrightarrow{v} Y'}{X \xrightarrow{\bar{g}} GY \xrightarrow{Gv} GY'}$$

Proposition. $\mathbb{C}$ has binary products if & only if the diagonal functor

$$\Delta = \langle id, id \rangle : \mathbb{C} \to \mathbb{C} \times \mathbb{C}$$

has a right adjoint.

Proposition A cartesian category $\mathbb{C}$ has all exponentials if & only if for all $X \in \mathrm{Obj}\,\mathbb{C}$, the functor

$$(-) \times X : \mathbb{C} \to \mathbb{C}$$

has a right adjoint.