

This lecture

Some category theory relevant
to modelling type theories with
dependent types

Will restrict attention to
what it looks like just in **Set**
rather ~~than~~ in full generality

[See e.g. : M. Hofmann, "Syntax & Semantics of Dependent Types", pp 79-130 of A. Pitts & P. Dybjer, "Semantics & Logics of Computation (CUP, 1997)]

Simple types

$$x_1 : T_1, \dots, x_n : T_n \vdash t(x_1, \dots, x_n) : T$$

Simple types

$$x_1 : T_1, \dots, x_n : T_n \vdash t(x_1, \dots, x_n) : T$$

Dependent types

$$x_1 : T_1, \dots, x_n : T_n \vdash t(x_1, \dots, x_n) : T(x_1, \dots, x_n)$$

& more generally

$$x_1 : T_1, x_2 : T_2(x_1), x_3 : T_3(x_1, x_2), \dots \vdash t : T(x_1, x_2, \dots)$$

If types denote sets,
then a dependent type

$$T(x) \quad [x : T']$$

should denote an

indexed family of sets

$$E = (E_i \mid i \in I)$$

i.e. E is a function $I \rightarrow \text{obj}(\text{Set})$

For each $I \in \text{obj Set}$, let $\boxed{\text{Set}^I}$ be the category with

- $\text{obj}(\text{Set}^I) \triangleq (\text{obj Set})^I$
so objects are I -indexed families of sets
 $X = (X_i \mid i \in I)$
- morphisms $f: X \rightarrow Y$ in Set^I are
 I -indexed families of functions
 $f = (f_i \in \text{Set}(X_i, Y_i) \mid i \in I)$

- composition: $(g \circ f) = (g_i \circ f_i \mid i \in I)$
identities: $\text{id}_X = (\text{id}_{X_i} \mid i \in I)$


For each $p: I \rightarrow J$ in Set , let

$$p^*: \text{Set}^J \rightarrow \text{Set}^I$$

be the functor

$$p^* \left(\begin{array}{c} Y \\ \downarrow f \\ Y' \end{array} \right) \triangleq \left(\begin{array}{c} Y_{p(i)} \\ \downarrow f_{p(i)} \\ Y'_{p(i)} \end{array} \mid i \in I \right)$$

i.e. p^* takes J -indexed families of sets/functions to I -indexed ones by precomposing with p

Dependent products of families of sets

for $I, J \in \text{Obj Set}$, projection $\pi_1: I \times J \rightarrow I$
gives a functor $\pi_1^*: \text{Set}^I \rightarrow \text{Set}^{I \times J}$

Theorem π_1^* has a left adjoint
 $\Sigma: \text{Set}^{I \times J} \rightarrow \text{Set}^I$

Proof For each $E \in \text{Set}^{I \times J}$ we give

$$\Sigma E \in \text{Set}^I \quad \& \quad \eta_E: E \rightarrow \pi_1^*(\Sigma E)$$

with required universal property ...

For each $E \in \text{Set}^{I \times J}$, we define $\Sigma E \in \text{Set}^I$ by:

$$(\Sigma E)_i \triangleq \sum_{j \in J} E_{(i,j)} = \{(j, e) \mid j \in J \wedge e \in E_{(i,j)}\}$$

(all $i \in I$)

and $\eta_E \in \text{Set}^{I \times J}(E, \pi_1^*(\Sigma E))$ by

$$(\eta_E)_{(i,j)} : E_{(i,j)} \rightarrow (\Sigma E)_i$$
$$e \mapsto (j, e)$$

(all $(i,j) \in I \times J$)

Universal property of $\eta_E : E \rightarrow \pi_1^*(\Sigma E) :$

Given any $X \in \text{Set}^I$ & $f : E \rightarrow \pi_1^*(X)$
in $\text{Set}^{I \times J}$, we have:

existence

$$\begin{array}{ccc}
 E & \xrightarrow{\eta_E} & \pi_1^*(\Sigma E) \\
 & \searrow f & \downarrow \pi_1^*(\hat{f}) \\
 & & \pi_1^*(X)
 \end{array}$$

$$\begin{array}{c}
 \Sigma E \\
 \downarrow \hat{f} \\
 X
 \end{array}$$

← in Set^I

where

$$\hat{f}_i(j, e) \triangleq f_{(i,j)}(e) \quad \left(\text{all } \begin{array}{l} i \in I \\ j \in J \\ e \in E_{(i,j)} \end{array} \right)$$

Universal property of $\eta_E : E \rightarrow \pi_1^*(\Sigma E)$

Given any $X \in \text{Set}^I$ & $f : E \rightarrow \pi_1^*(X)$
in $\text{Set}^{I \times J}$, we have:

uniqueness

$$\begin{array}{ccc}
 E & \xrightarrow{\eta_E} & \pi_1^*(\Sigma E) \\
 & \searrow f & \downarrow \pi_1^*(g) \\
 & & \pi_1^*(X)
 \end{array}$$

if f

$$\begin{array}{c}
 \Sigma E \\
 \downarrow g \\
 X
 \end{array}$$

← in Set^I

then for all $i \in I$, $(j, e) \in (\Sigma E)_i$, have

$$\hat{f}_i(j, e) = f_{(ij)}(e) = (\pi_1^* g \circ \eta_E)_{(ij)} e = g_i(j, e) \quad \text{So } g = \hat{f} \quad \square$$

Dependent functions for families of sets

for $I, J \in \text{obj Set}$, projection $\pi_1: I \times J \rightarrow I$
gives a functor $\pi_1^*: \text{Set}^I \rightarrow \text{Set}^{I \times J}$

Theorem π_1^* has a right adjoint
 $\Pi: \text{Set}^{I \times J} \rightarrow \text{Set}^I$

Proof For each $E \in \text{Set}^{I \times J}$ we give

$$\Pi E \in \text{Set}^I \quad \& \quad \varepsilon_E: \pi_1^*(\Pi E) \rightarrow E$$

with required universal property ...

For each $E \in \text{Set}^{I \times J}$, we define $\prod E \in \text{Set}^I$
by:

$$(\prod E)_i \triangleq \prod_{j \in J} E_{(i,j)}$$

$$= \{ f \subseteq (\sum E)_i \mid f \text{ is single-valued \& total} \}$$

where $f \subseteq (\sum E)_i$ is

single-valued if $(\forall j \in J)(\forall e, e' \in E_{(i,j)})(j, e) \in f \wedge (j, e') \in f \Rightarrow e = e'$

total if $(\forall j \in J)(\exists e \in E_{(i,j)})(j, e) \in f$

ie. each $f \in (\prod E)_i$ is a dependently typed
function mapping elements $j \in J$ to elements
of $E_{(i,j)}$ depends on argument i

For each $E \in \text{Set}^{I \times J}$, we define $\Pi E \in \text{Set}^I$

by:

$$(\Pi E)_i \triangleq \prod_{j \in J} E_{(i,j)}$$

$$= \{ f \subseteq (\Sigma E)_i \mid f \text{ is single-valued \& total} \}$$

and $\varepsilon_E \in \text{Set}^{I \times J} (\pi_1^*(\Pi E), E)$ by

$$(\varepsilon_E)_{(i,j)} : (\Pi E)_i \rightarrow E_{(i,j)}$$

$$f \longmapsto f(j)$$

(all $(i,j) \in I \times J$)

the unique
 $e \in E_{(i,j)}$ such
 that $(j, e) \in f$

Universal property of $\varepsilon_E : \pi_1^*(\pi E) \rightarrow E$

Given any $X \in \text{Set}^I$ and $f : \pi_1^*(X) \rightarrow E$ in $\text{Set}^{I \times J}$, we have:

existence

$$\begin{array}{c} \pi E \\ \hat{f} \uparrow \\ X \end{array}$$

$$\begin{array}{ccc} \pi_1^*(\pi E) & \xrightarrow{\varepsilon_E} & E \\ \pi_1^*(\hat{f}) \uparrow & \nearrow f & \\ \pi_1^*(X) & & \end{array}$$

in Set^I

where $\hat{f}_i(x) \triangleq \{(j, f_{(i,j)}(x)) \mid j \in J\}$
(all $i \in I, x \in X_i$)

Universal property of $\varepsilon_E : \pi_1^*(\pi E) \rightarrow E$

Given any $X \in \text{Set}^I$ and $f : \pi_1^*(X) \rightarrow E$ in $\text{Set}^{I \times J}$, we have :

uniqueness

if

$$\begin{array}{c} \pi E \\ \uparrow g \\ X \end{array}$$

$$\begin{array}{ccc} \pi_1^*(\pi E) & \xrightarrow{\varepsilon_E} & E \\ \pi_1^*(g) \uparrow & \nearrow f & \\ \pi_1^*(X) & & \end{array}$$

in Set^I

then $\hat{f}_i x_j = f_{(i,j)} x = (\varepsilon_E \circ \pi_1^* g)_{(i,j)} x = g_i x_j$

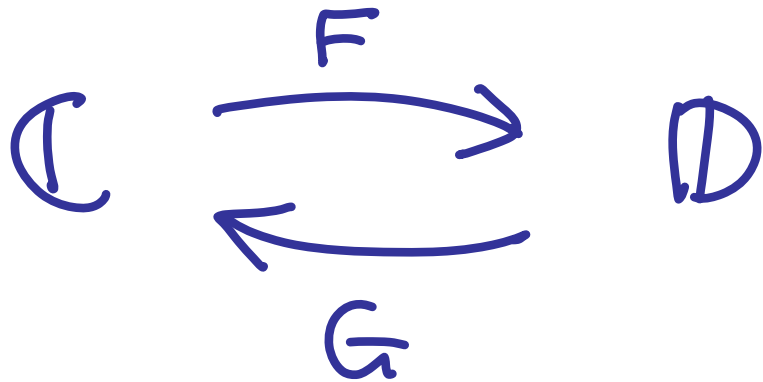
so $g = \hat{f}$



Equivalence

Two categories $\mathbb{C}$ & $\mathbb{D}$ are **isomorphic** if they are isomorphic objects in the category of categories (of some size), that is, there are

functors



satisfying

$$\begin{aligned} \text{Id}_{\mathbb{C}} &= G \circ F \\ F \circ G &= \text{Id}_{\mathbb{D}} \end{aligned}$$

(in which case, as usual, we write $\mathbb{C} \cong \mathbb{D}$)

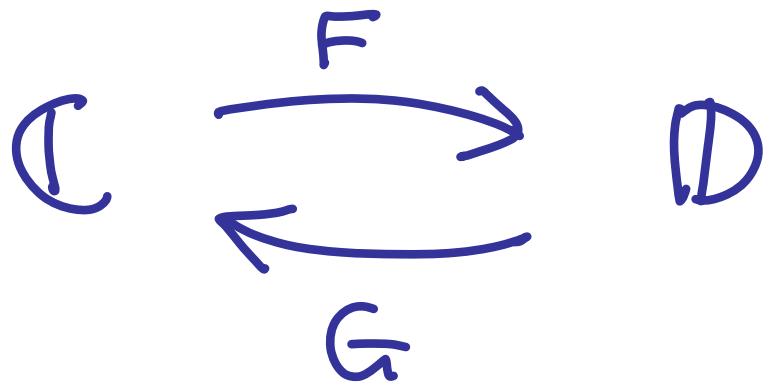
Equivalence

Two categories $\mathcal{C}$ & $\mathcal{D}$ are **equivalent** if there are

functors

&

natural isomorphisms



$$\eta : \text{Id}_{\mathcal{C}} \cong G \circ F$$

$$\varepsilon : F \circ G \cong \text{Id}_{\mathcal{D}}$$

in which case one writes

$$\mathcal{C} \cong \mathcal{D}$$

Example: $\text{Set}^I \simeq \text{Set}/I$

Set/I is a *slice category* [Ex. Sh.4, qu.6]

- objects are (x, f) where $f \in \text{Set}(x, I)$
- morphisms $g : (x, f) \rightarrow (x', f')$ are $g \in \text{Set}(x, x')$ satisfying $f' \circ g = f$ in Set
- composition & identities - as for Set

Example: $\text{Set}^I \simeq \text{Set}/I$

functor $F : \text{Set}^I \rightarrow \text{Set}/I$

on objects: $F(X) \triangleq \left(\begin{array}{c} \{(i, x) \mid i \in I \ \& \ x \in X_i\} \\ \downarrow \text{fst} \\ I \end{array} \right)$

on morphisms:

$F(X \xrightarrow{f} X') \triangleq$

$$\begin{array}{ccc} \{(i, x) \mid i \in I \ \& \ x \in X_i\} & \rightarrow & \{(i, x') \mid i \in I \ \& \ x' \in X'_i\} \\ & \searrow & \swarrow \\ & I & \\ (i, x) & \xrightarrow{\quad} & (i, f_i x) \end{array}$$

Example: $\text{Set}^I \simeq \text{Set}/I$

functor $G: \text{Set}/I \rightarrow \text{Set}^I$

on objects: $G\left(\begin{array}{c} E \\ \downarrow p \\ I \end{array}\right) \triangleq \left(\{e \in E \mid pe = i\} \mid i \in I \right)$

on morphisms:

$G\left(\begin{array}{ccc} E & \xrightarrow{f} & E' \\ p \searrow & & \swarrow p' \\ & I & \end{array}\right) \triangleq G E \xrightarrow{Gf} G E' \text{ where for}$
each $i \in I$
 $(Gf)_i e \triangleq f(e)$

Example : $\text{Set}^I \cong \text{Set}/I$

Exercise : Construct natural isomorphisms

$$\eta : \text{Id}_{\text{Set}^I} \cong G \circ F$$

$$\varepsilon : F \circ G \cong \text{Id}_{\text{Set}/I}$$

to show that $\text{Set}^I \cong \text{Set}/I$.

FACT Given $p: I \rightarrow J$ in $\mathbf{Set}$,

$$\mathbf{Set}/J \simeq \mathbf{Set}^J \xrightarrow{p^*} \mathbf{Set}^I \simeq \mathbf{Set}/I$$

is the functor "**pullback** along p "

Can generalize from $\mathbf{Set}$ to any
category $\mathcal{C}$ with pullbacks & model
 Σ/Π types by left/right adjoints
to pullback functor - see **locally cartesian**
closed categories in literature.