Exercise Sheet 1 available on course web page — answers next week

Office hours: Wednesdays 12-1pm
FC08

After today, lectures take place in
FS07

Definition

A **category** $\mathbb{C}$ is specified by

- a collection $\mathrm{Obj}\,\mathbb{C}$ of **$\mathbb{C}$-objects** $X, Y, Z, \ldots$
- for each $X, Y \in \mathrm{Obj}\,\mathbb{C}$, a collection $\mathbb{C}(X, Y)$ of **$\mathbb{C}$-morphisms from X to Y**
- an operation assigning to each $X \in \mathrm{Obj}\,\mathbb{C}$, an **identity morphism** $id_X \in \mathbb{C}(X, X)$
- an operation assigning to each $f \in \mathbb{C}(X, Y)$ & $g \in \mathbb{C}(Y, Z)$ a **composition** $g \circ f \in \mathbb{C}(X, Z)$

satisfying ...

Definition, cont.

Satisfying ...

Associativity: for all $f \in \mathbb{C}(X,Y)$, $g \in \mathbb{C}(Y,Z)$ & $h \in \mathbb{C}(Z,W)$

$$\boxed{h \circ (g \circ f) = (h \circ g) \circ f}$$

Unity: for all $f \in \mathbb{C}(X,Y)$

$$\boxed{id_Y \circ f = f = f \circ id_X}$$

Example: category of pre-orders $\mathbf{Pre}$

- objects are sets with a pre-order

$(P, \_\leq\_)$

$\boxed{P \in \mathrm{Set}}$

$\_\leq\_ \subseteq P \times P$ is a binary relation which is

<u>reflexive</u>: $(\forall x \in P)\ x \leq x$

<u>transitive</u>: $(\forall x, y, z \in P)\ x \leq y \wedge y \leq z \Rightarrow x \leq z$

(a partial order is a pre-order that is also anti-symmetric: $(\forall x, y \in P)\ x \leq y \wedge y \leq x \Rightarrow x = y$)

Example: category of pre-orders $\mathbf{Pre}$

- objects are sets with a pre-order
- morphisms: $\mathbf{Pre}((P, \leq), (Q, \leq))$

$$\triangleq \{ f \in \mathbf{Set}(P, Q) \mid f \text{ is monotone} \}$$

$$\boxed{(\forall x, x' \in P)\ x \leq x' \Rightarrow fx \leq fx'}$$

- identities & composition as for $\mathbf{Set}$

(why does this make sense ?)

Example: category of pre-orders $\mathbf{Pre}$

- objects are sets with a pre-order

Pre-orders are relevant to
denotational semantics of prog. langs.
(among other things)

Example pre-order

$$(X \rightharpoonup Y, \subseteq)$$

inclusion

set of partial functions
from X to Y

Example: category of monoids, **Mon**

- objects are monoids

$$(M, \_\bullet\_, e)$$

$\boxed{M \in \mathbf{Set}}$

$\boxed{e \in M}$

$\_\bullet\_ \in \mathbf{Set}(M \times M, M)$
binary operation which is
<u>associative</u> $(\forall x, y, z \in M)$
$x \bullet (y \bullet z) = (x \bullet y) \bullet z$
<u>has e as unit</u>
$(\forall x \in M)\ e \bullet x = x = x \bullet e$

Example: category of monoids, **Mon**

- objects are monoids
- morphisms $\mathbf{Mon}((M,\cdot,e),(M',\cdot',e'))$

$$\triangleq \{ f \in \mathbf{Set}(M,M') \mid f \text{ is a homomorphism of monoids} \}$$

$$\boxed{\begin{array}{l} f\,e = e' \;\;\& \\ (\forall x,y \in M)\; f(x\cdot y) = (f x)\cdot'(f y) \end{array}}$$

Example: category of monoids, $\mathbf{Mon}$

- objects are monoids
- morphisms $\mathbf{Mon}((M, \cdot, e), (N, \cdot', e')) \triangleq \{ f \in \mathbf{Set}(M, N) \mid f$ is a homomorphism of monoids $\}$
- identities & composition as for $\mathbf{Set}$

(why does this make sense?)

Example: category of monoids, Mon

- objects are monoids

  Monoids are relevant to

  automata theory

  (among other things)

Example monoid:

$(\mathrm{List}(\Sigma), @, \mathrm{nil})$

list concatenation

empty list

set of all finite lists over a set Σ

Example: every pre-order $(P, \leq)$ is a category

- objects = elements of P
- morphisms

$$P(x,y) \triangleq \begin{cases} \{*\} & \text{if } x \leq y \\ \emptyset & \text{if } x \not\leq y \end{cases}$$

a one-element set

empty set

- identities & composition ... are uniquely determined (why?)

Example: every monoid $(M, \cdot, e)$ is a category

- just one object (call it $*$)
- $M(*,*) \triangleq M$
- $id_* \triangleq e$ (monoid unit element)
- composition of $f \in M(*,*)$ & $g \in M(*,*)$ is $g \circ f = g \cdot f$ (monoid binary op^n)

Some finite categories

empty

$$\begin{tikzcd} \bullet \arrow[loop above, "id"] \end{tikzcd}$$

one object, one morphism

$$\begin{tikzcd} 0 \arrow[r, "f"] \arrow[loop left, "id_0"] & 1 \arrow[loop right, "id_1"] \end{tikzcd}$$

two objects, one non-identity morphism

Definition of isomorphism

Let $\mathbb{C}$ be a category. A $\mathbb{C}$-morphism $f \in \mathbb{C}(X,Y)$ is an isomorphism if there is some $g \in \mathbb{C}(Y,X)$ with

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[dr, bend right, "id_X"'] & Y \arrow[d, "g"] \arrow[dr, bend left, "id_Y"] & \\
& X \arrow[r, "f"'] & Y
\end{tikzcd}
$$

Definition of isomorphism

$$\boxed{\text{Let } \mathbb{C} \text{ be a category. A } \mathbb{C}\text{-morphism } f \in \mathbb{C}(X,Y) \text{ is an isomorphism if there is some } g \in \mathbb{C}(Y,X) \text{ with } g \circ f = id_X \ \& \ f \circ g = id_Y}$$

- Such a g is uniquely determined by f (why?) and we write $\boxed{f^{-1}}$ for g.
- Given $X, Y \in \mathbb{C}$, if such an f exists, we say X & Y are isomorphic objects and write $\boxed{X \cong Y}$.

Theorem $f \in \mathrm{Set}(X, Y)$ is an isomorphism iff f is a bijection, that is,

injective $(\forall x, x' \in X)\, fx = fx' \Rightarrow x = x'$

&

surjective $(\forall y \in Y)(\exists x \in X)\, fx = y$

Proof ...

$\boxed{\text{if \& only if}}$

Theorem $f \in \mathrm{Mon}((M,\cdot,1),(N,\cdot,1))$ is an isomorphism iff $f \in \mathrm{Set}(M,N)$ is a bijection.

Proof ...

Define **Pos** to be the category whose objects are posets (= pre-ordered sets for which the pre-order is anti-symmetric) & whose morphisms are monotone functions. (identities & composition as for **Pre**)

Theorem $f \in \mathrm{Pos}((P,\leqslant),(Q,\leqslant))$ is an isomorphism iff $f \in \mathrm{Set}(P,Q)$ is surjective and reflects the partial order, that is

$$(\forall p,p' \in P)\ fp \leqslant fp' \Rightarrow p \leqslant p'$$

Proof ...

(why does this not work for Pre ?)

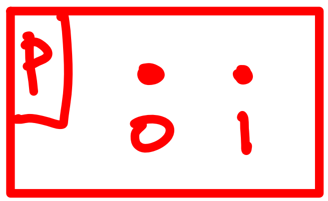
Theorem $f \in \mathbf{Pos}((P,\leq),(Q,\leq))$ is an isomorphism iff $f \in \mathbf{Set}(P,Q)$ is surjective and reflects the partial order, that is

$$(\forall p, p' \in P)\ fp \leq fp' \Rightarrow p \leq p'$$

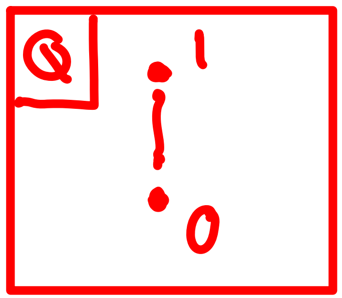
Example to show that $P \cong Q$ in $\mathbf{Set}$ does not necessarily imply $(P,\leq) \cong (Q,\leq)$ in $\mathbf{Pos}$.

Take $P = Q = \{0,1\}$

$\leq$ on P to be $\{(0,0),(1,1)\}$



$\leq$ on Q to be $\{(0,0),(0,1),(1,1)\}$



$(P,\leq) \not\cong (Q,\leq)$ (why?)