

L108 Assessment heads up

Assessed exercise sheet (ExSh#4)
(for 25% credit)

- issued Monday 9 Nov (in class)
- your answers are due by
Monday 16 Nov, 16:00

(Take-home exam, 75% credit, in Jan.)

I -ary products

Given a set I and an I -indexed family $(X_i \mid i \in I)$ of objects X_i of a category $\mathcal{C}$, their product is an object $\prod_{i \in I} X_i$ plus morphisms $\pi_i : (\prod_{i \in I} X_i) \rightarrow X_i$ (for each $i \in I$) with the universal property...

with the universal property...

for all $(f_i : Z \rightarrow X_i \mid i \in I)$

there exists a unique

$$\langle f_i \mid i \in I \rangle : Z \rightarrow \prod_{i \in I} X_i$$

such that

$$\pi_j \circ \langle f_i \mid i \in I \rangle = f_j$$

for all $j \in I$

Case $I = \emptyset$; terminal object !

Case $I = \{0, 1\}$; binary product

Case $I = \{1, \dots, n\}$ often written $X_1 \times \dots \times X_n$

Cartesian categories

are categories that possess I -ary products for all finite set I ,

or equivalently,

that have a terminal object and all binary products.

Examples: Set , Pos , Mon , meet semi-lattice, ...
poset with glbs for all finite subsets ↗

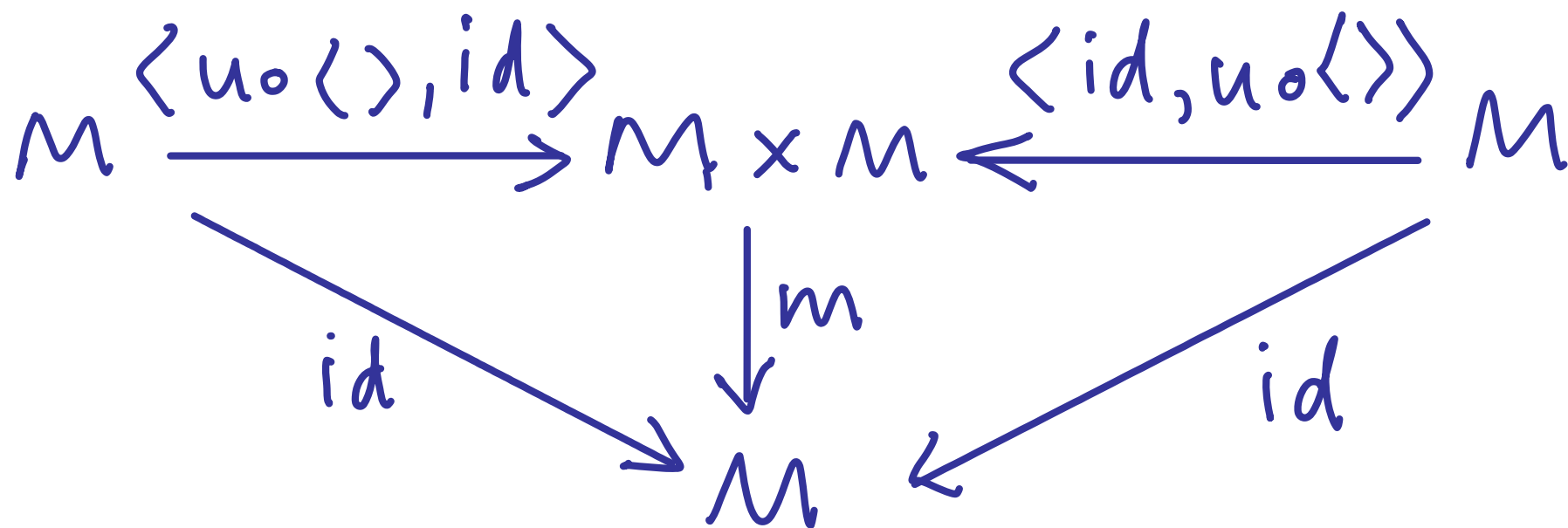
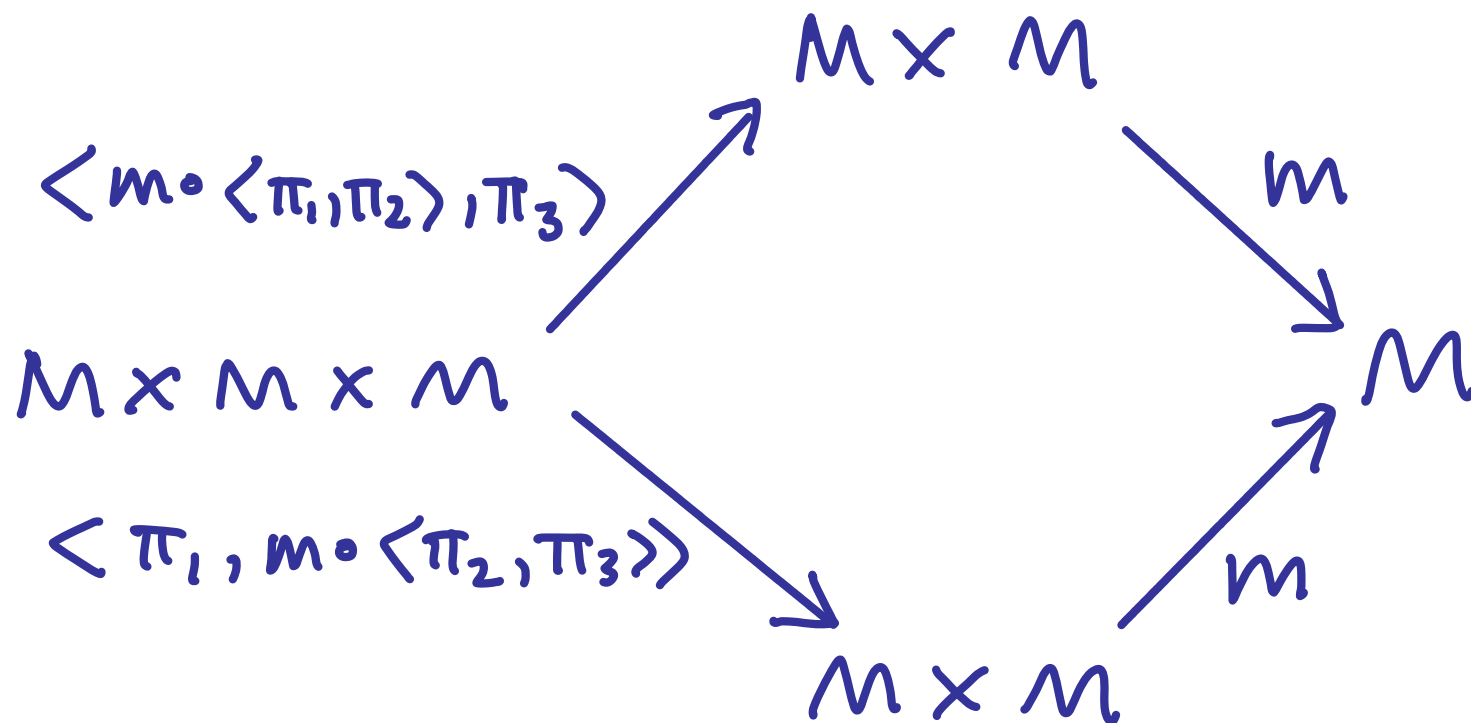
Algebraic structure, categorically

E.g. Can describe a monoid $(M, \cdot, e)$ as:

- object M in $\mathbf{Set}$
- morphisms $\begin{cases} m : M \times M \rightarrow M \\ u : 1 \rightarrow M \end{cases}$ in $\mathbf{Set}$

Such that the following diagrams commute in $\mathbf{Set}$

note that there's a bijection
 $M \cong \mathbf{Set}(1, M)$
 $m \mapsto \ulcorner m \urcorner$ where $\ulcorner m \urcorner(0) = m$
 $f(0) \longleftrightarrow f$



Algebraic structure, categorically

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- morphisms $\begin{cases} m : M \times M \rightarrow M \\ u : 1 \rightarrow M \end{cases}$

Such that the following diagrams commute ...

↖ makes sense in any cartesian category $\mathcal{C}$
E.g. a monoid in $\mathbf{Pos}$ is ...

Algebraic Signatures, Σ

are specified by:

- a collection of sorts $S_1, S_2, \dots$
- a collection of operation symbols $f, g, \dots$ each of which comes with a typing

$$f : [S_1, \dots, S_n] \rightarrow S$$

the argument type
— a finite list of sorts

the result type
of f — a sort

the arity of f

Arity

= number of arguments an operation symbol takes

$n=0$ nullary opⁿ. Symbol is called a constant

$n=1$ unary

$n=2$ binary

$n=3$ ternary

⋮

n n-ary

E.g. signature for monoids

Sort $*$
operation symbols $\left\{ \begin{array}{l} m : [* , *] \rightarrow * \\ u : [] \rightarrow * \end{array} \right.$

E.g. a signature for Boolean algebras
"with contradictions"

Sorts B, C
opⁿ. symbols $\left\{ \begin{array}{l} \text{true} : [] \rightarrow B \\ \text{false} : [] \rightarrow B \\ \text{or} : [B, B] \rightarrow B \\ \text{not} : [B] \rightarrow B \\ \text{contr} : [C] \rightarrow B \end{array} \right.$

(see Ex. Sheet 2, qu. 7)

Algebraic terms over a signature Σ

Fix a countably infinite set
of variables $x, y, z, \dots$

N.B. when
 $n=0$, often
write $f()$
just as f

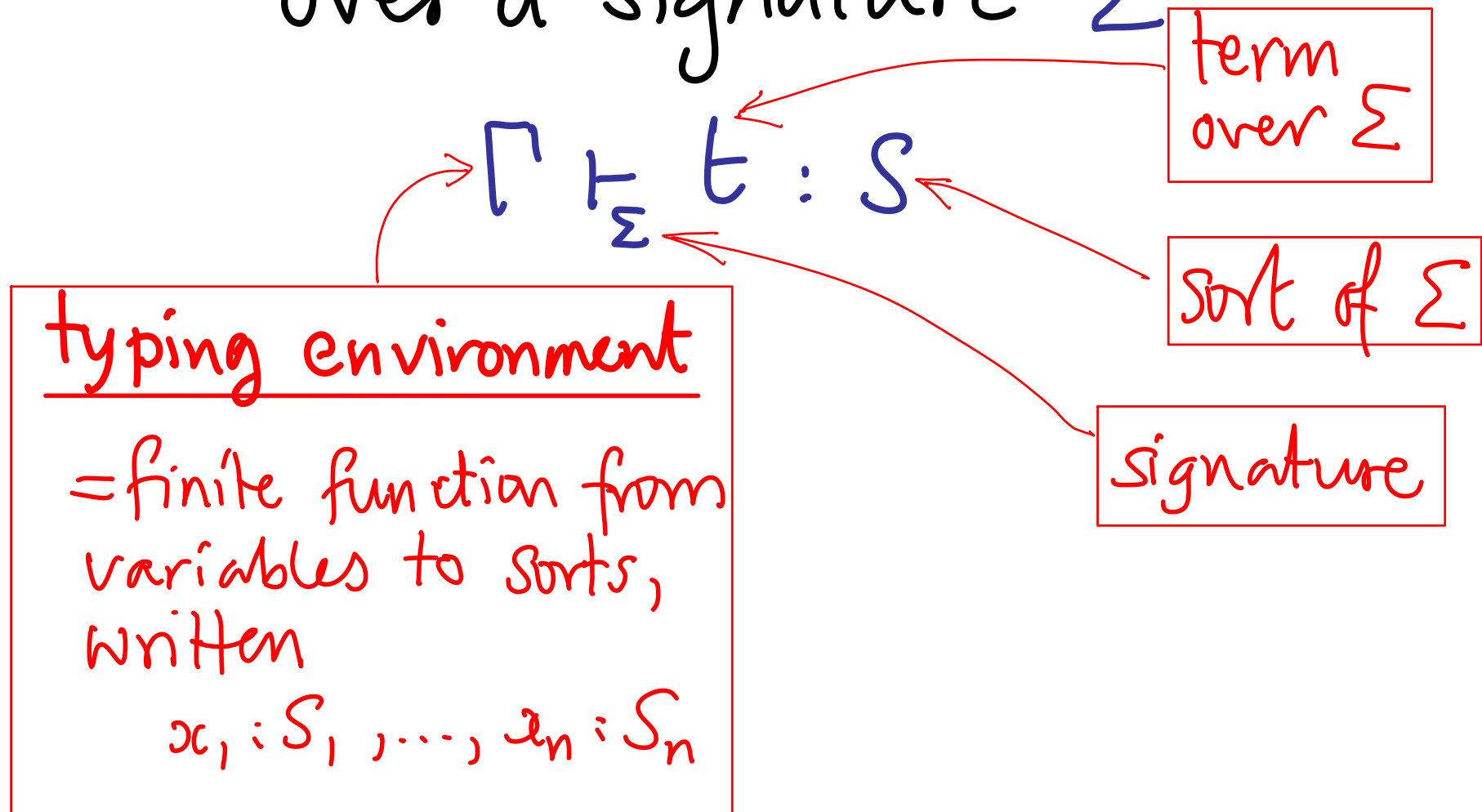
Terms :

$$t ::= x \mid f(t_1, \dots, t_n)$$

variable

operation symbol of the
signature of arity n

Typing relation over a signature Σ



is inductively generated by the rules...

Typing judgement over a signature Σ

is inductively generated by the rules...

$$\frac{(x : S) \in \Gamma}{\Gamma \vdash_{\Sigma} x : S}$$

$$\frac{\Gamma \vdash_{\Sigma} t_1 : S_1 \dots \Gamma \vdash_{\Sigma} t_n : S_n \quad (f : [S_1, \dots, S_n] \rightarrow S) \in \Sigma}{\Gamma \vdash_{\Sigma} f(t_1, \dots, t_n) : S}$$

Structures

A **structure** for an algebraic signature Σ in a cartesian category $\mathbb{C}$ is given by:

- for each sort S , a $\mathbb{C}$ -object $\llbracket S \rrbracket$
- for each operation symbol $f: [S_1, \dots, S_n] \rightarrow S$, a $\mathbb{C}$ -morphism

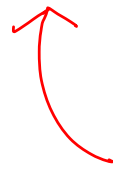
$$\llbracket f \rrbracket: \llbracket S_1 \rrbracket \times \dots \times \llbracket S_n \rrbracket \rightarrow \llbracket S \rrbracket$$

 product in $\mathbb{C}$ of the objects $\llbracket S_1 \rrbracket, \dots, \llbracket S_n \rrbracket$

Structures

A **structure** for an algebraic signature Σ in a cartesian category $\mathcal{C}$ allows us to interpret each valid typing judgement $\Gamma \vdash_\Sigma t : S$ as a $\mathcal{C}$ -morphism

$$\llbracket t \rrbracket : \llbracket \Gamma \rrbracket \rightarrow \llbracket S \rrbracket$$



$$\begin{aligned} \llbracket \Gamma \rrbracket &\triangleq \llbracket S_1 \rrbracket \times \dots \times \llbracket S_n \rrbracket \\ \text{if } \Gamma &\text{ is } x_1 : S_1, \dots, x_n : S \end{aligned}$$

Structures

A **structure** for an algebraic signature Σ in a cartesian category $\mathcal{C}$ allows us to interpret each valid typing judgement $\Gamma \vdash_\Sigma t : S$ as a $\mathcal{C}$ -morphism

$$\llbracket t \rrbracket : \llbracket \Gamma \rrbracket \longrightarrow \llbracket S \rrbracket$$

$$\llbracket x \rrbracket = \pi_i : S_1 \times \dots \times S_n \rightarrow S_i \quad \text{if } x = x_i$$

$$\llbracket f(t_1, \dots, t_n) \rrbracket = \llbracket f \rrbracket \circ \langle \llbracket t_1 \rrbracket, \dots, \llbracket t_n \rrbracket \rangle$$

Structures

E.g. 'if Σ = Signature for monoids
and we have a structure for it in $\mathbb{C}$ with
 $\llbracket * \rrbracket = M$ and $\llbracket m \rrbracket = f: M \times M \rightarrow M$,
then from

$$x:*, y:*, z:* \vdash_{\Sigma} m(x, m(y, z)):*$$

we get $\mathbb{C}$ -morphism

$$\begin{aligned} \llbracket m(x, m(y, z)) \rrbracket &= f \circ \langle \pi_1, f \circ \langle \pi_2, \pi_3 \rangle \rangle \\ &: M \times M \times M \rightarrow M \end{aligned}$$