L108 Assessment heads up

Assessed exercise sheet (ExSh#4)
(for 25% credit)

- issued Monday 9 Nov (in class)
- your answers are due by Monday 16 Nov, 16:00

(Take-home exam, 75% credit, in Jan.)

I-ary products

Given a set I and an I-indexed family $(X_i \mid i \in I)$ of objects X_i of a category $\mathbb{C}$, their product is an object $\prod_{i\in I} X_i$ plus morphisms $\pi_i : (\prod_{i\in I} X_i) \longrightarrow X_i$ (for each $i \in I$) with the universal property...

with the universal property...

for all $(f_i : Z \to X_i \mid i \in I)$

there exists a unique

$\langle f_i \mid i \in I \rangle : Z \to \prod_{i \in I} X_i$

such that

$\pi_j \circ \langle f_i \mid i \in I \rangle = f_j$

for all $j \in I$

Case $I = \emptyset$; terminal object!

Case $I = \{0,1\}$; binary product

Case $I = \{1, \ldots, n\}$ often written $X_1 \times \cdots \times X_n$

Cartesian categories

are categories that possess I-ary products for all <u>finite</u> set I,

or equivalently,

that have a terminal object and all binary products.

Examples: **Set**, **Pos**, **Mon**, meet semi-lattice, ...

poset with glbs for all finite subsets

Algebraic structure, categorically

E.g. can describe a monoid $(M, \cdot, e)$ as:

- object M in $\mathbf{Set}$
- morphisms $\begin{cases} m : M \times M \to M \\ u : 1 \to M \end{cases}$ in $\mathbf{Set}$

such that the following diagrams commute in $\mathbf{Set}$

note that there's a bijection

$M \cong \mathbf{Set}(1, M)$

$m \longmapsto \ulcorner m \urcorner$ where $\ulcorner m \urcorner(0) = m$

$f(0) \longleftarrow\!\shortmid f$

$$
\begin{tikzcd}
 & M \times M \arrow[dr, "m"] & \\
M \times M \times M \arrow[ur, "\langle m \circ \langle \pi_1, \pi_2 \rangle, \pi_3 \rangle"] \arrow[dr, "\langle \pi_1, m \circ \langle \pi_2, \pi_3 \rangle \rangle"'] & & M \\
 & M \times M \arrow[ur, "m"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
M \arrow[r, "\langle u \circ \langle \rangle, id \rangle"] \arrow[dr, "id"'] & M \times M \arrow[d, "m"] & M \arrow[l, "\langle id, u \circ \langle \rangle \rangle"'] \arrow[dl, "id"] \\
 & M &
\end{tikzcd}
$$

Algebraic structure, categorically

E.g. can describe a monoid $(M, \cdot, 1)$ as:

- object M
- morphisms $\begin{cases} m : M \times M \to M \\ u : 1 \to M \end{cases}$

such that the following diagrams commute...

makes sense in <u>any</u> cartesian category $\mathbb{C}$

E.g. a monoid in Pos is...

Algebraic Signatures, Σ

are specified by:

- a collection of sorts $S_1, S_2, \ldots$
- a collection of operation symbols $f, g, \ldots$ each of which comes with a typing

$$f : [\underbrace{S_1, \ldots, S_n}_{\text{the argument type}}] \to \underbrace{S}_{\text{the result type of } f}$$

the argument type – a finite list of sorts

the result type of f – a sort

the arity of f

Arity

= number of arguments an operation symbol takes

$n=0$ nullary op^n. symbol is called a constant

$n=1$ unary

$n=2$ binary

$n=3$ ternary

$\vdots$

n n-ary

E.g. signature for monoids

Sort $*$

operation symbols $\begin{cases} m : [*,*] \to * \\ u : [\,] \to * \end{cases}$

E.g. a signature for Boolean algebras "with contradictions"

Sorts B, C

op^n. symbols $\begin{cases} \text{true} : [\,] \to B \\ \text{false} : [\,] \to B \\ \text{or} : [B,B] \to B \\ \text{not} : [B] \to B \\ \text{contr} : [C] \to B \end{cases}$

(see Ex. Sheet 2, qu. 7)

Algebraic terms
over a signature Σ

Fix a countably infinite set of variables $x, y, z, \ldots$

Terms:

$$t ::= x \mid f(t_1, \ldots, t_n)$$

variable

operation symbol of the signature of arity n

N.B. when $n = 0$, often write $f(\,)$ just as f

Typing relation

over a signature Σ

$$\Gamma \vdash_{\Sigma} t : S$$

term over Σ

sort of Σ

signature

typing environment

= finite function from variables to sorts, written

$x_1 : S_1, \ldots, x_n : S_n$

is inductively generated by the rules...

Typing judgement

over a signature Σ

is inductively generated by the rules...

$$\frac{(x : S) \in \Gamma}{\Gamma \vdash_\Sigma x : S}$$

$$\frac{\Gamma \vdash_\Sigma t_1 : S_1 \quad \cdots \quad \Gamma \vdash_\Sigma t_n : S_n \quad (f : [S_1, \ldots, S_n] \to S) \in \Sigma}{\Gamma \vdash_\Sigma f(t_1, \ldots, t_n) : S}$$

Structures

A **structure** for an algebraic signature Σ in a cartesian category $\mathbb{C}$ is given by:

- for each sort S, a $\mathbb{C}$-object $[\![S]\!]$
- for each operation symbol $f : [S_1, \ldots, S_n] \to S$, a $\mathbb{C}$-morphism

$$[\![f]\!] : \underbrace{[\![S_1]\!] \times \cdots \times [\![S_n]\!]} \longrightarrow [\![S]\!]$$

product in $\mathbb{C}$ of the objects $[\![S_1]\!], \ldots, [\![S_n]\!]$

Structures

A **structure** for an algebraic signature Σ in a cartesian category $\mathbb{C}$ allows us to interpret each valid typing judgement $\Gamma \vdash_\Sigma t : S$ as a $\mathbb{C}$-morphism

$$[\![t]\!] : [\![\Gamma]\!] \longrightarrow [\![S]\!]$$

$$\boxed{\begin{array}{l} [\![\Gamma]\!] \triangleq [\![S_1]\!] \times \cdots \times [\![S_n]\!] \\ \text{if } \Gamma \text{ is } x_1 : S_1, \ldots, x_n : S \end{array}}$$

Structures

A **structure** for an algebraic signature Σ in a cartesian category $\mathbb{C}$ allows us to interpret each valid typing judgement $\Gamma \vdash_\Sigma t : S$ as a $\mathbb{C}$-morphism

$$[\![t]\!] : [\![\Gamma]\!] \longrightarrow [\![S]\!]$$

$$\boxed{\begin{array}{c} [\![x]\!] = \pi_i : S_1 \times \cdots \times S_n \to S_i \quad \text{if } x = x_i \\ \hline [\![f(t_1, \ldots, t_n)]\!] = [\![f]\!] \circ \langle [\![t_1]\!], \ldots, [\![t_n]\!] \rangle \end{array}}$$

Structures

E.g. if Σ = signature for monoids
and we have a structure for it in $\mathbb{C}$ with
$[\![*]\!] = M$ and $[\![m]\!] = f : M \times M \to M$,
then from

$$x : *, y : *, z : * \vdash_{\Sigma} m(x, m(y,z)) : *$$

we get $\mathbb{C}$-morphism

$$[\![m(x, m(y, z))]\!] = f \circ \langle \pi_1, f \circ \langle \pi_2, \pi_3 \rangle\rangle$$
$$: M \times M \times M \longrightarrow M$$