Recall

Algebraic terms over a signature Σ

Fix a countably infinite set of variables $x, y, z, \cdots$

N.B. when $n=0$, often write $f(\,)$ just as f

Terms:

$$t ::= x \mid f(t_1, \ldots, t_n)$$

variable

operation symbol of the signature of arity n

Substitution

The substituted term $\boxed{t'[t/x]}$ is defined by recursion on the structure of the term t':

- if t' is a variable, then

$$y[t/x] = \begin{cases} t & \text{if } y = x \\ y & \text{otherwise} \end{cases}$$

- if t' is a compound term, then

$$f(t'_1, \ldots, t'_m)[t/x] = f(t'_1[t/x], \ldots, t'_m[t/x])$$

Simultaneous substitution

$t'[t/x]$ is a special case of

$t'[t_1/x_1, \ldots, t_n/x_n]$ (where $x_1, \ldots, x_n$ are distinct)

defined by:

- $y[t_1/x_1, \ldots, t_n/x_n] = \begin{cases} t_i & \text{if } y = x_i \\ y & \text{if } y \notin \{x_1, \ldots, x_n\} \end{cases}$
- $f(t'_1, \ldots, t'_m)[t_1/x_1, \ldots] = f(t'_1[t_1/x_1, \ldots], \ldots, t'_m[t_1/x_1, \ldots])$

Recall **Typing judgement** over a signature Σ

is inductively generated by the rules...

$$\boxed{\dfrac{(x : S) \in \Gamma}{\Gamma \vdash_\Sigma x : S}}$$

$$\boxed{\dfrac{\Gamma \vdash_\Sigma t_1 : S_1 \quad \cdots \quad \Gamma \vdash_\Sigma t_n : S_n \quad (f : [S_1, \ldots, S_n] \to S) \in \Sigma}{\Gamma \vdash_\Sigma f(t_1, \ldots, t_n) : S}}$$

Lemma If $\Gamma \vdash_\Sigma t_1 : S_1, \ldots, \Gamma \vdash_\Sigma t_n : S_n$ and $x_1 : S_1, \ldots, x_n : S_n \vdash_\Sigma t' : S'$, then $\Gamma \vdash_\Sigma t'[t_1/x_1, \ldots, t_n/x_n] : S'$

Proof by induction on the structure of t'.

Substitution lemma For any structure for an alg. sig. in a cartesian category:

$$[\![t'[t_1/x_1, \ldots, t_n/x_n]]\!] = [\![t']\!] \circ \langle [\![t_1]\!], \ldots, [\![t_n]\!] \rangle$$

Proof by induction on the structure of t'.

Recall

Structures

A structure for an algebraic signature Σ in a cartesian category $\mathbb{C}$ allows us to interpret each valid typing judgement $\Gamma \vdash_\Sigma t : S$ as a $\mathbb{C}$-morphism

$$[\![t]\!] : [\![\Gamma]\!] \longrightarrow [\![S]\!]$$

$$[\![x]\!] = \pi_i : S_1 \times \cdots \times S_n \to S_i \quad \text{if } x = x_i$$

$$[\![f(t_1, \ldots, t_n)]\!] = [\![f]\!] \circ \langle [\![t_1]\!], \ldots, [\![t_n]\!] \rangle$$

Equations

over a signature Σ

take the form $\boxed{\Gamma \vdash_\Sigma t = t' : S}$

where $\Gamma \vdash_\Sigma t : S$ and $\Gamma \vdash_\Sigma t' : S$.

Algebraic theory $= \begin{cases} \text{algebraic signature} \\ + \\ \text{set of equations} \\ \text{(the theory's axioms)} \end{cases}$

E.g. alg. theory of monoids has equations

$x:*, y:*, z:* \vdash m(x, m(y,z)) = m(m(x,y),z) : *$	
$x:* \vdash m(u(), x) = x : *$	$x:* \vdash m(x, u()) = x : *$

Example: an algebraic theory of lists

sorts: V (values) L (lists of values)

operation symbols:

$nil : [\,] \to L$

$cons : [V, L] \to L$

$hd : [L] \to V$

$tl : [L] \to L$

$apnd : [L, L] \to L$

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axioms:

$$x : V, \ell : L \vdash \mathrm{hd}(\mathrm{cons}(x, \ell)) = x : V$$

$$x : V, \ell : L \vdash \mathrm{tl}(\mathrm{cons}(x, \ell)) = \ell : L$$

$$\ell : L \vdash \mathrm{apnd}(\mathrm{nil}, \ell) = \ell : L$$

$$x : V, \ell : L, \ell' : L \vdash \mathrm{apnd}(\mathrm{cons}(x, \ell), \ell') = \mathrm{cons}(x, \mathrm{apnd}(\ell, \ell')) : L$$

Equational Logic

The **theorems** of an algebraic theory consist of all the equations derivable from its axioms using the following rules...

$$\frac{\Gamma \vdash_\Sigma t : S}{\Gamma \vdash_\Sigma t = t : S} \qquad \frac{\Gamma \vdash_\Sigma t = t' : S}{\Gamma \vdash_\Sigma t' = t : S} \qquad \frac{\Gamma \vdash_\Sigma t = t' : S \quad \Gamma \vdash_\Sigma t' = t'' : S}{\Gamma \vdash_\Sigma t = t'' : S}$$

$$\frac{\Gamma \vdash_\Sigma t_1 = t_1' : S_1 \quad \cdots \quad \Gamma \vdash_\Sigma t_n = t_n' : S_n \qquad x_1 : S_1, \ldots, x_n : S_n \vdash_\Sigma t = t' : S}{\Gamma \vdash_\Sigma t[t_1/x_1, \ldots, t_n/x_n] = t'[t_1'/x_1, \ldots, t_n'/x_n] : S}$$

Satisfaction

A structure in a cartesian category for an algebraic signature **satisfies** an equation $\Gamma \vdash_\Sigma t = t' : S$ if

$$[\![t]\!] = [\![t']\!] : [\![\Gamma]\!] \to [\![S]\!]$$

<u>Soundness Theorem</u> If a structure satisfies all the axioms of an algebraic theory, then it satisfies all its theorems

<u>Proof</u> Just have to check that satisfaction is closed under the rules of equational logic
– for the substitution rule, use the substitution lemma.

Algebras

An algebra for an algebraic theory $\mathbb{T}$ in a cartesian category $\mathbb{C}$ is a structure that satisfies all the axioms of $\mathbb{T}$

[There's a category of $\mathbb{T}$-algebras in $\mathbb{C}$, the morphism of which are algebra homomorphisms (definition omitted).]

The internal language

of a cartesian category $\mathbb{C}$ is the algebraic signature with

- one sort for each $\mathbb{C}$-object
- one operation symbol $f : [X_1, \ldots, X_n] \to X$ for each non-empty list $[X_1, \ldots, X_n, X]$ of $\mathbb{C}$-objects and each $\mathbb{C}$-morphism $f : X_1 \times \cdots \times X_n \longrightarrow X$

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→ there's a structure in $\mathbb{C}$ for this signature

namely $\begin{cases} [\![X]\!] = X \\ [\![f]\!] = f \end{cases}$

Terms & equations over this signature allow us to describe properties in $\mathbb{C}$, for example...

Proposition

the given terminal object of $\mathbb{C}$

(1) $X \in \mathbb{C}$ is a terminal object in $\mathbb{C}$ iff there is some $\mathbb{C}$-morphism $t : 1 \to X$ satisfying (in the internal lang. of $\mathbb{C}$)

$$\boxed{x : X \vdash x = t() : X}$$

"T has just one element"

(2) $X \xleftarrow{p} Z \xrightarrow{q} Y$ is a product in $\mathbb{C}$ iff there is a $\mathbb{C}$-morphism $r : X \times Y \to Z$ satisfying

$$x : X, y : Y \vdash p(r(x,y)) = x : X$$
$$x : X, y : Y \vdash q(r(x,y)) = y : Y$$
$$z : Z \vdash z = r(p(z), q(z)) : Z$$

Theories as categories

From an algebraic theory $\mathbb{T}$, can construct a cartesian category $\mathbb{C}_{\mathbb{T}}$:

- objects are typing contexts $\Gamma = x_1 : S_1, \ldots, x_n : S_n$
- morphisms $\Gamma \to \Gamma'$ are equivalence classes $[t_1', \ldots, t_m']$ where $\Gamma' = x_1' : S_1', \ldots, x_m' : S_m'$ & $\Gamma \vdash t_i' : S_i$ $(i = 1, \ldots m)$ for equiv. relation given by theorem of $\mathbb{T}$
- composition given by substitution, identities given by variables

Theories as categories

From an algebraic theory $\mathbb{T}$, can construct a cartesian category $\mathbb{C}_{\mathbb{T}}$ – see Section 4.2 of [Pitts, Categorical Logic].

There's a structure for $\mathbb{T}$ in $\mathbb{C}_{\mathbb{T}}$ that satisfies all its axioms

and $\Gamma \vdash t = t' : S$ is a theorem of $\mathbb{T}$ iff it's satisfied by this structure.

[Hence cartesian categories are complete for equational logic]