Non-example of a ccc

Category of monoids **Mon** is <u>not</u> a ccc,
because:

free monoid on $2 = \{0,1\}$

by univ. prop. of free monoid

$$\mathbb{N} \cong 2^* \times 2^* \cong \mathbf{Set}(2, 2^*) \cong \mathbf{Mon}(2^*, 2^*)$$

because $1 \times M \cong M$

$$\cong \mathbf{Mon}(1 \times 2^*, 2^*)$$

(Here I'm writing X^* instead of List(X) for the set of finite lists of elements of a set X.)

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so $\mathbf{Mon}(1 \times 2^*, 2^*) \not\cong \mathbf{Mon}(1, M)$ for any M, and hence the exponential of 2^* & 2^* can't exist in **Mon**.

Examples of ccc's

A pre-ordered set $(X, \leq)$ regarded as a category is cartesian iff it has

- a greatest element $\top$: $(\forall p \in P)\ p \leq \top$
- binary meets $p \wedge q$: $(\forall r \in P)\ r \leq p \wedge q \iff r \leq p \ \&\ r \leq q$

It is a ccc iff it has

- Heyting implications $p \to q$: $(\forall r \in P)\ r \leq p \to q \iff r \wedge p \leq q$

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E.g. any Boolean algebra ($p \to q = \neg p \vee q$),
Also $([0,1], \leq)$, for which $p \to q = \begin{cases} 1 & \text{if } p \leq q \\ q & \text{if } q < p \end{cases}$

Intuitionistic Propositional Logic

- "natural deduction" style
- only conjunction & implication fragment

Formulas:

$$\varphi, \psi, \theta, \ldots \;::=\; p, q, r, \ldots \qquad \text{propositional identifiers}$$

$$\top \qquad \text{truth}$$

$$\varphi \,\&\, \psi \qquad \text{conjunction}$$

$$\varphi \Rightarrow \psi \qquad \text{implication}$$

Intuitionistic Propositional Logic

Entailment relation $\Phi \vdash \varphi$

hypotheses,
a finite multiset
(= unordered list)
of formulas

conclusion,
a formula

is inductively defined by the following rules:

Intuitionistic Propositional Logic

$$\frac{\Phi \vdash \varphi \qquad \Phi, \varphi \vdash \psi}{\Phi \vdash \psi}\,(\mathrm{Cut})$$

$$\frac{}{\Phi, \varphi \vdash \varphi}\,(\mathrm{Ax})$$

$$\frac{}{\Phi \vdash \top}\,(\top)$$

$$\frac{\Phi \vdash \varphi \qquad \Phi \vdash \psi}{\Phi \vdash \varphi \,\&\, \psi}\,(\wedge I)$$

$$\frac{\Phi, \varphi \vdash \psi}{\Phi \vdash \varphi \Rightarrow \psi}\,(\Rightarrow I)$$

$$\frac{\Phi \vdash \varphi \,\&\, \psi}{\Phi \vdash \varphi}\,(\wedge E_1)$$

$$\frac{\Phi \vdash \varphi \,\&\, \psi}{\Phi \vdash \psi}\,(\wedge E_2)$$

$$\frac{\Phi \vdash \varphi \Rightarrow \psi \qquad \Phi \vdash \varphi}{\Phi \vdash \psi}\,(\Rightarrow E)$$

For example $\varphi \Rightarrow \psi, \psi \Rightarrow \theta \vdash \varphi \Rightarrow \theta$ holds :

$$\dfrac{\overbrace{\boxed{\varphi \Rightarrow \psi, \psi \Rightarrow \theta, \varphi}}^{\Phi} \vdash \theta}{\varphi \Rightarrow \psi, \psi \Rightarrow \theta \vdash \varphi \Rightarrow \theta}\ (\Rightarrow I)$$

For example $\varphi \Rightarrow \psi, \psi \Rightarrow \theta \vdash \varphi \Rightarrow \theta$ holds :

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$$\left(\Phi \triangleq \varphi \Rightarrow \psi, \psi \Rightarrow \theta, \varphi \right)$$

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$$
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$$

$$(\Phi \triangleq \varphi \Rightarrow \psi, \psi \Rightarrow \theta, \varphi)$$

Semantics of IPL in a cartesian closed pre-order $(P, \leq)$

Given a meaning for each propositional identifier p as an element $[\![p]\!] \in P$, we get a semantics for formulas $[\![\varphi]\!] \in P$:

$$[\![\top]\!] = 1$$ greatest element

$$[\![\varphi \& \psi]\!] = [\![\varphi]\!] \wedge [\![\psi]\!]$$ binary meet

$$[\![\varphi \Rightarrow \psi]\!] = [\![\varphi]\!] \to [\![\psi]\!]$$ Heyting implication

Semantics of IPL in a cartesian closed pre-order $(P, \leq)$

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and a semantics for multisets of formulas

$[\![\Phi]\!] \in P$:

$$[\![\varnothing]\!] = 1$$

$$[\![\Phi, \varphi]\!] = [\![\Phi]\!] \wedge [\![\varphi]\!]$$

Semantics of IPL in a cartesian closed pre-order $(P, \leq)$

Soundness theorem

If $\Phi \vdash \varphi$ is provable from the rules of IPL, then $[\![\Phi]\!] \leq [\![\varphi]\!]$ holds in any cartesian closed pre-order.

Proof – exercise.

Example

application of the Soundness Theorem :

> **Peirce's Law** $\nvdash ((\varphi \Rightarrow \psi) \Rightarrow \varphi) \Rightarrow \varphi$ is <u>not</u> provable in IPL

(whereas $((\varphi \Rightarrow \psi) \Rightarrow \varphi) \Rightarrow \varphi$ is a classical tautology)

because in the c.c. pre-order $([0,1], \leq)$ taking $[\![\varphi]\!] = \frac{1}{2}$, $[\![\psi]\!] = 0$ we get

$$
\begin{aligned}
[\![((\varphi \Rightarrow \psi) \Rightarrow \varphi) \Rightarrow \varphi]\!] &= ((\tfrac{1}{2} \to 0) \to \tfrac{1}{2}) \to \tfrac{1}{2} \\
&= (0 \to \tfrac{1}{2}) \to \tfrac{1}{2} \\
&= 1 \to \tfrac{1}{2} \\
&= \tfrac{1}{2}
\end{aligned}
$$

Semantics of IPL in a cartesian closed poset $(P, \leq)$

<u>Completeness Theorem</u>

If $[\![\Phi]\!] \leq [\![\varphi]\!]$ holds in all c.c pre-orders then $\Phi \vdash \varphi$ is provable in IPL.

<u>Proof</u> ...

Proof Define

$$\boxed{\begin{array}{l} P \triangleq \{\text{formulas of IPL}\} \\ \varphi \leq \psi \triangleq \{\varphi\} \vdash \psi \end{array}}$$

Then $(P, \leq)$ is a c.c. pre-ordered set with an interpretation of IPL given by $[\![p]\!] = p$.

Can show that $[\![\Phi]\!] \leq [\![\psi]\!]$ in this $(P, \leq)$ iff $\Phi \vdash \psi$ is valid in IPL.

$\square$