L108 Assessment heads up

Assessed exercise sheet (Ex Sh #4)
(for 25% credit)

- issued Monday 9 Nov (in class)
- your answers are due by Monday 16 Nov, 16:00

(Take-home exam, 75% credit, in Jan.)

Recall the derivation of $\varphi \Rightarrow \psi, \psi \Rightarrow \theta \vdash \varphi \Rightarrow \theta$:

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{(Ax)}
\UnaryInfC{$\Phi \vdash \psi \Rightarrow \theta$}
\AxiomC{}
\RightLabel{(Ax)}
\UnaryInfC{$\Phi \vdash \varphi \Rightarrow \psi$}
\AxiomC{}
\RightLabel{(Ax)}
\UnaryInfC{$\Phi \vdash \varphi$}
\RightLabel{$(\Rightarrow E)$}
\BinaryInfC{$\Phi \vdash \psi$}
\RightLabel{$(\Rightarrow E)$}
\BinaryInfC{$\Phi \vdash \theta$}
\RightLabel{$(\Rightarrow I)$}
\UnaryInfC{$\varphi \Rightarrow \psi, \psi \Rightarrow \theta \vdash \varphi \Rightarrow \theta$}
\end{prooftree}
$$

$$\left(\Phi \triangleq \varphi \Rightarrow \psi, \psi \Rightarrow \theta, \varphi \right)$$

Another derivation of $\varphi \Rightarrow \psi, \psi \Rightarrow \theta \vdash \varphi \Rightarrow \theta$:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Phi \vdash \varphi \Rightarrow \psi}(Ax) \quad \dfrac{}{\Phi \vdash \varphi}(Ax)}{\Phi \vdash \psi}(\Rightarrow E) \qquad \dfrac{\dfrac{}{\Phi, \psi \vdash \psi \Rightarrow \theta}(Ax) \quad \dfrac{}{\Phi, \psi \vdash \psi}(Ax)}{\Phi, \psi \vdash \theta}(\Rightarrow E)}{\Phi \vdash \theta}(Cut)}{\varphi \Rightarrow \psi, \psi \Rightarrow \theta \vdash \varphi \Rightarrow \theta}(\Rightarrow I)$$

$$\left(\Phi \triangleq \varphi \Rightarrow \psi, \psi \Rightarrow \theta, \varphi \right)$$

Proof Theory

$$
\dfrac{\dfrac{}{\Phi \vdash \psi \Rightarrow \theta}\,(\mathrm{Ax}) \qquad \dfrac{\dfrac{}{\Phi \vdash \varphi \Rightarrow \psi}\,(\mathrm{Ax}) \quad \dfrac{}{\Phi \vdash \varphi}\,(\mathrm{Ax})}{\Phi \vdash \psi}\,(\Rightarrow E)}{\dfrac{\Phi \vdash \theta}{\varphi \Rightarrow \psi, \psi \Rightarrow \theta \vdash \varphi \Rightarrow \theta}\,(\Rightarrow I)}\,(\Rightarrow E)
$$

$(\Phi \triangleq \varphi \Rightarrow \psi, \psi \Rightarrow \theta, \varphi)$

Why is this proof simpler than this one?

$$
\dfrac{\dfrac{\dfrac{}{\Phi \vdash \varphi \Rightarrow \psi}\,(\mathrm{Ax}) \quad \dfrac{}{\Phi \vdash \varphi}\,(\mathrm{Ax})}{\Phi \vdash \psi}\,(\Rightarrow E) \qquad \dfrac{\dfrac{}{\Phi, \psi \vdash \psi \Rightarrow \theta}\,(\mathrm{Ax}) \quad \dfrac{}{\Phi, \psi \vdash \psi}\,(\mathrm{Ax})}{\Phi, \psi \vdash \theta}\,(\Rightarrow E)}{\dfrac{\Phi \vdash \theta}{\varphi \Rightarrow \psi, \psi \Rightarrow \theta \vdash \varphi \Rightarrow \theta}\,(\Rightarrow I)}\,(\mathrm{Cut})
$$

$(\Phi \triangleq \varphi \Rightarrow \psi, \psi \Rightarrow \theta, \varphi)$

FACT: if $\Phi \vdash \varphi$ is derivable, it is derivable WITHOUT USING THE CUT RULE ("cut elimination")

Proof Theory

$$
\dfrac{
  \dfrac{}{\Phi \vdash \psi \Rightarrow \theta}\,(\text{Ax})
  \qquad
  \dfrac{
    \dfrac{}{\Phi \vdash \varphi \Rightarrow \psi}\,(\text{Ax})
    \quad
    \dfrac{}{\Phi \vdash \varphi}\,(\text{Ax})
  }{\Phi \vdash \psi}\,(\Rightarrow E)
}{
  \dfrac{\Phi \vdash \theta}{\varphi \Rightarrow \psi, \psi \Rightarrow \theta \vdash \varphi \Rightarrow \theta}\,(\Rightarrow I)
}\,(\Rightarrow E)
$$

$(\Phi \triangleq \varphi \Rightarrow \psi, \psi \Rightarrow \theta, \varphi)$

Why is this proof simpler than this one?

$$
\dfrac{
  \dfrac{
    \dfrac{}{\Phi \vdash \varphi \Rightarrow \psi}\,(\text{Ax})
    \quad
    \dfrac{}{\Phi \vdash \varphi}\,(\text{Ax})
  }{\Phi \vdash \psi}\,(\Rightarrow E)
  \qquad
  \dfrac{
    \dfrac{}{\Phi, \psi \vdash \psi \Rightarrow \theta}\,(\text{Ax})
    \quad
    \dfrac{}{\Phi, \psi \vdash \psi}\,(\text{Ax})
  }{\Phi, \psi \vdash \theta}\,(\Rightarrow E)
}{
  \dfrac{\Phi \vdash \theta}{\varphi \Rightarrow \psi, \psi \Rightarrow \theta \vdash \varphi \Rightarrow \theta}\,(\Rightarrow I)
}\,(\text{Cut})
$$

$(\Phi \triangleq \varphi \Rightarrow \psi, \psi \Rightarrow \theta, \varphi)$

Need a language & calculus of proofs [for IPL] to answer questions like this...

Simply Typed Lambda Calculus (STLC)
(with finite products)

Simple types:

$$A, B, C, \ldots \quad ::= \quad G, G', \ldots \qquad \text{“ground” types}$$
$$\mid\quad 1 \qquad \text{unit type}$$
$$\mid\quad A \times B \qquad \text{product type}$$
$$\mid\quad A \to B \qquad \text{function type}$$

Simply Typed Lambda Calculus (STLC)
(with finite products)

Terms:

$s, t, r, \ldots \; ::= \; c^A$

x

$()$

(s, t)

$\mathrm{fst}\; t$

$\mathrm{snd}\; t$

$\lambda x{:}A.t$

$s\,t$

Constants
each with
a given type

Variables
(countably
many)

λ-abstraction

application

Simple types:

$A, B, C, \ldots \; ::= \; G, G', \ldots$

1

$A \times B$

$A \to B$

Alpha Equivalence

STLC terms are abstract syntax trees modulo renaming λ-bound variables.

E.g. $\lambda f : A \to B . \lambda x : A . f x$

& $\lambda x : A \to B . \lambda y : A . x y$

are the same term.

Alpha Equivalence

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Formally, we quotient syntax trees by the equivalence relation of α-equivalence $=_\alpha$

(or use a "nameless" (de Bruijn) representation).

Alpha Equivalence

$$\frac{}{c^A =_\alpha c^A} \qquad \frac{}{x =_\alpha x} \qquad \frac{}{() =_\alpha ()}$$

$$\frac{s =_\alpha s' \quad t =_\alpha t'}{(s,t) =_\alpha (s',t')} \qquad \frac{t =_\alpha t'}{\mathrm{fst}\, t =_\alpha \mathrm{fst}\, t'} \qquad \frac{t =_\alpha t'}{\mathrm{snd}\, t =_\alpha \mathrm{snd}\, t'}$$

$$\frac{s =_\alpha s' \quad t =_\alpha t'}{s\,t =_\alpha s'\,t'}$$

result of replacing all occurrences of x with y in term t

$$\frac{(y\ x)\cdot t =_\alpha (y\ x')\cdot t' \quad y \text{ does not occur in } \{x, x', t, t'\}}{\lambda x{:}A.\,t =_\alpha \lambda x'{:}A.\,t'}$$

Simply Typed Lambda Calculus (STLC)

Typing relation $\Gamma \vdash t : A$

typing environment
= finite function from variables to types, written
$[x_1 : A_1, \ldots, x_n : A_n]$
($x_1, \ldots, x_n$ distinct, $A_1, \ldots, A_n$ not necessarily distinct)

term

type

is inductively defined by the following rules...

$$\frac{}{\Gamma \vdash c^A : A}\ (\text{const})$$

$$\frac{(x : A) \in \Gamma}{\Gamma \vdash x : A}\ (\text{var})$$

$$\frac{}{\Gamma \vdash () : 1}\ (\text{unit})$$

$$\frac{\Gamma \vdash s : A \quad \Gamma \vdash t : B}{\Gamma \vdash (s,t) : A \times B}\ (\text{pair})$$

$$\frac{\Gamma \vdash t : A \times B}{\Gamma \vdash \mathrm{fst}\ t : A}\ (\text{fst})$$

$$\frac{\Gamma \vdash t : A \times B}{\Gamma \vdash \mathrm{snd}\ t : B}\ (\text{snd})$$

$$\frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x{:}A.t : A \to B}\ (\lambda)$$

$$\frac{\Gamma \vdash s : A \to B \quad \Gamma \vdash t : A}{\Gamma \vdash s\,t : B}\ (\text{app})$$

N.B. $\Gamma, x : A$ means… (can & do assume $x \notin \Gamma$)

Example typing derivation

$$
\dfrac{
  \dfrac{}{\Gamma, x : A \vdash g : B \to C}\,(\text{var})
  \qquad
  \dfrac{
    \dfrac{}{\Gamma, x : A \vdash f : A \to B}\,(\text{var})
    \quad
    \dfrac{}{\Gamma, x : A \vdash x : A}\,(\text{var})
  }{\Gamma, x : A \vdash f x : B}\,(\text{app})
}{
  \dfrac{\Gamma, x : A \vdash g(f x) : C}{\Gamma \vdash \lambda x : A .\, g(f x) : A \to C}\,(\lambda)
}\,(\text{app})
$$

(where $\Gamma \triangleq [f : A \to B,\ g : B \to C]$)

<u>N.B.</u> typing rules are "syntax-directed" (by the structure of t)

Semantics of STLC types in a ccc $\mathbb{C}$

Given a function

ground types $G \mapsto$ objects $[\![G]\!] \in \mathbb{C}$

we extend it to a function

types $A \mapsto$ objects $[\![A]\!] \in \mathbb{C}$

by recursion on the structure of A:

$$\boxed{\begin{aligned} [\![1]\!] &= 1 \quad \text{terminal object} \\ [\![A \times B]\!] &= [\![A]\!] \times [\![B]\!] \quad \text{product} \\ [\![A \to B]\!] &= [\![B]\!]^{[\![A]\!]} \quad \text{exponential} \end{aligned}}$$

Semantics of STLC types in a ccc $\mathbb{C}$

$$\boxed{\begin{aligned} [\![1]\!] &= 1 \\ [\![A \times B]\!] &= [\![A]\!] \times [\![B]\!] \\ [\![A \to B]\!] &= [\![B]\!]^{[\![A]\!]} \end{aligned}}$$

extend this to typing environments:

$$[\![x_1 : A_1, \ldots, x_n : A_n]\!] \triangleq [\![A_1]\!] \times \cdots \times [\![A_n]\!]$$

n-fold product in $\mathbb{C}$

Semantics of STLC terms in a ccc $\mathbb{C}$

Given a function

constants $c^A \mapsto |c^A| \in \mathbb{C}(1, [\![A]\!])$

we get a function

provable typing $\Gamma \vdash t : A \mapsto [\![t]\!] \in \mathbb{C}\left([\![\Gamma]\!], [\![A]\!]\right)$

defined by recursion on the structure of t

as follows...

- $[\![c^A]\!] = [\![\Gamma]\!] \xrightarrow{\langle\rangle} 1 \xrightarrow{\ulcorner c^A \urcorner} [\![A]\!]$

- $[\![x_i]\!] = [\![\Gamma]\!] \xrightarrow{\pi_i} [\![A_i]\!]$ if $\Gamma = [\ldots, x_i : A_i, \ldots]$

- $[\![()]\!] = [\![\Gamma]\!] \xrightarrow{\langle\rangle} 1 = [\![1]\!]$

- $[\![(s,t)]\!] = [\![\Gamma]\!] \xrightarrow{\langle [\![s]\!], [\![t]\!] \rangle} [\![A]\!] \times [\![B]\!] = [\![A \times B]\!]$

- $[\![\mathrm{fst}\ t]\!] = [\![\Gamma]\!] \xrightarrow{[\![t]\!]} [\![A \times B]\!] = [\![A]\!] \times [\![B]\!] \xrightarrow{\pi_1} [\![A]\!]$
  $[\![\mathrm{snd}\ t]\!] = \pi_2 \circ [\![t]\!]$

- $[\![\lambda x : A . t]\!] = \mathrm{cur}\left([\![\Gamma]\!] \times [\![A]\!] \cong [\![\Gamma, x : A]\!] \xrightarrow{[\![t]\!]} [\![B]\!] \right)$

- $[\![s\,t]\!] = [\![\Gamma]\!] \xrightarrow{\langle [\![s]\!], [\![t]\!] \rangle} [\![B]\!]^{[\![A]\!]} \times [\![A]\!] \xrightarrow{\mathrm{app}} [\![B]\!]$

For example, consider

$$u : A \to B,\ v : B \to C \vdash \lambda x : A.\, v(ux) : A \to C$$

Suppose $[\![A]\!] = X$, $[\![B]\!] = Y$, $[\![C]\!] = Z$.

Then $\begin{cases} [\![u : A \to B, v : B \to C]\!] = Y^X \times Z^Y \\ [\![A \to C]\!] = Z^X \end{cases}$

and $[\![\lambda x : A.\, v(ux)]\!] : Y^X \times Z^Y \longrightarrow Z^X$

is the morphism

$$\mathrm{cur}\left((Y^X \times Z^Y) \times X \xrightarrow[\cong]{\langle \pi_1\pi_1, \pi_2\pi_1, \pi_2 \rangle} Y^X \times Z^Y \times X \xrightarrow{\langle \pi_2, \mathrm{app} \circ \langle \pi_1, \pi_3 \rangle \rangle} Z^Y \times Y \xrightarrow{\mathrm{app}} Z \right)$$