# 13: Betweenness Centrality

## Machine Learning and Real-world Data (MLRD)

Ann Copestake
(based on slides created by Simone Teufel)

Lent 2018

# Last session: some simple network statistics

- You measured the **degree** of each node and the **diameter** of the network.
- Next two sessions:
  - Today: finding **gatekeeper** nodes via **betweenness centrality**.
  - Next session: using betweenness centrality of edges to split graph into **cliques**.
- Reading for social networks (all sessions):
  - Easley and Kleinberg for background: Chapters 1, 2, 3 and first part of Chapter 20.
  - Brandes algorithm: two papers by Brandes (links in practical notes).

# Intuition behind clique finding

- Certain nodes/edges are most crucial in linking densely connected regions of the graph: informally **gatekeepers**.
- Cutting those edges isolates the cliques/clusters.

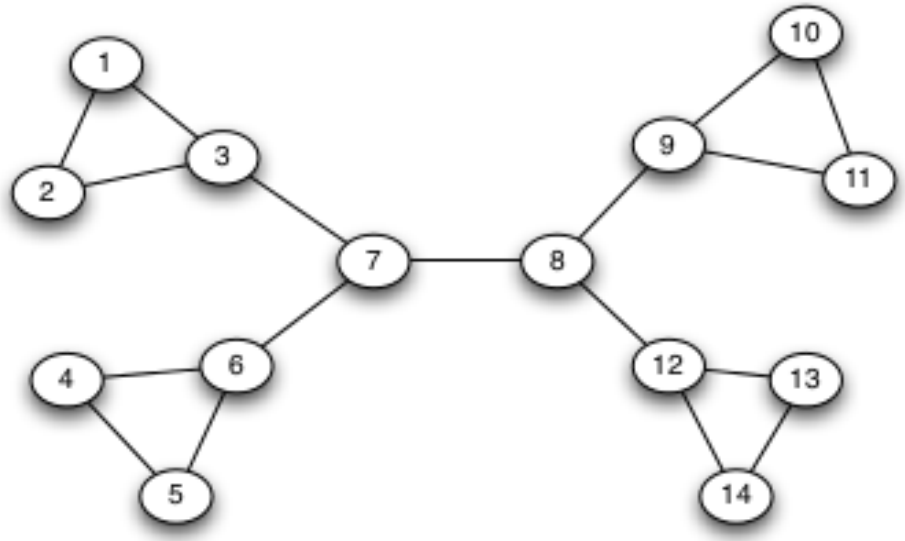



Figure 3-14a from Easley and Kleinberg (2010)

# Intuition behind clique finding

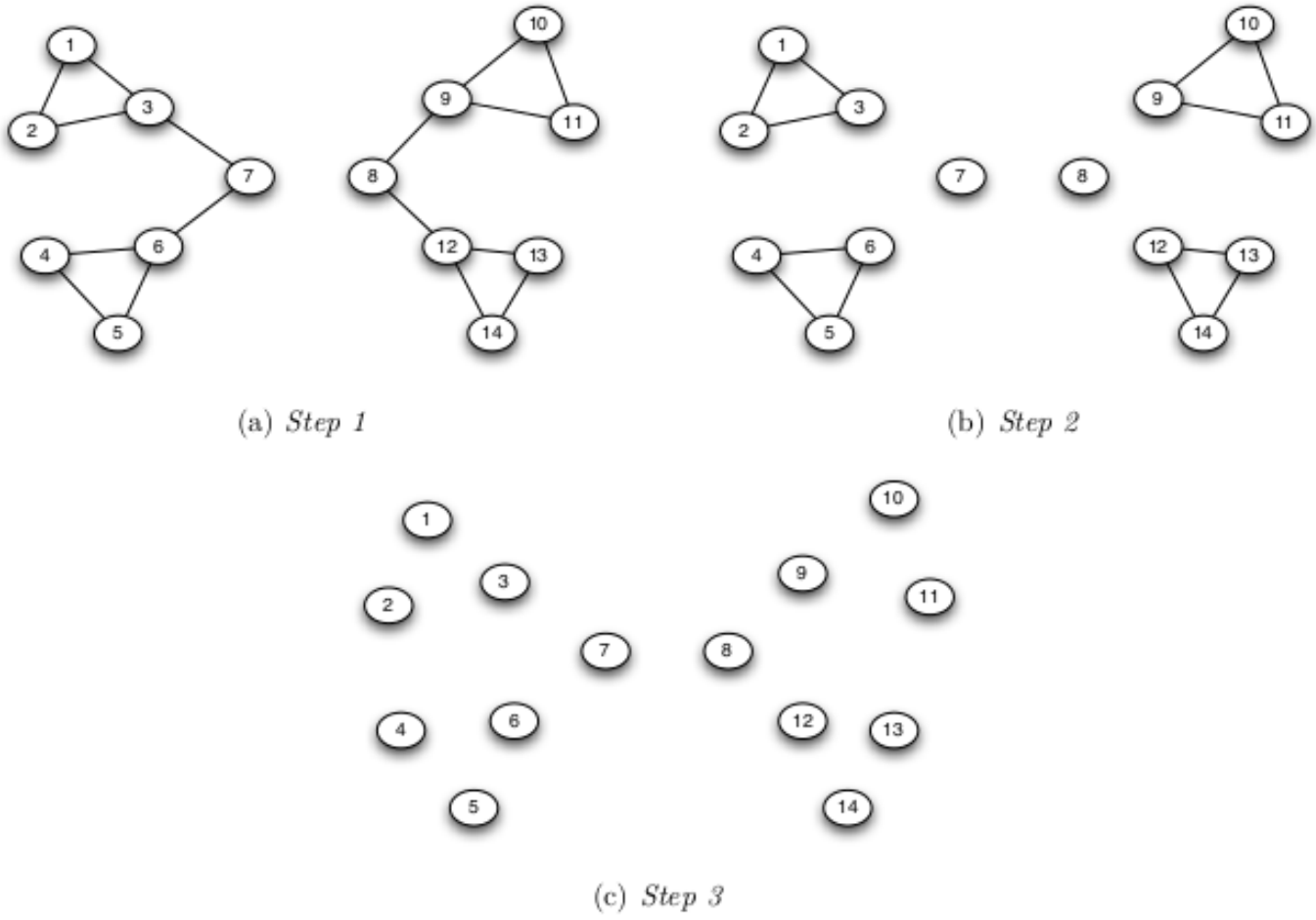



Figure 3-16 from Easley and Kleinberg (2010)

# Gatekeepers: generalising the notion of local bridge

- Last time we saw the concept of **local bridge**: an edge which increased the shortest paths if cut.

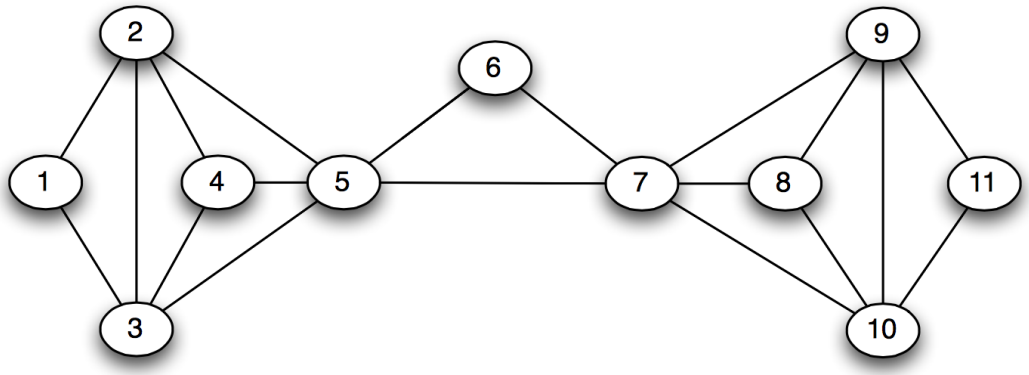
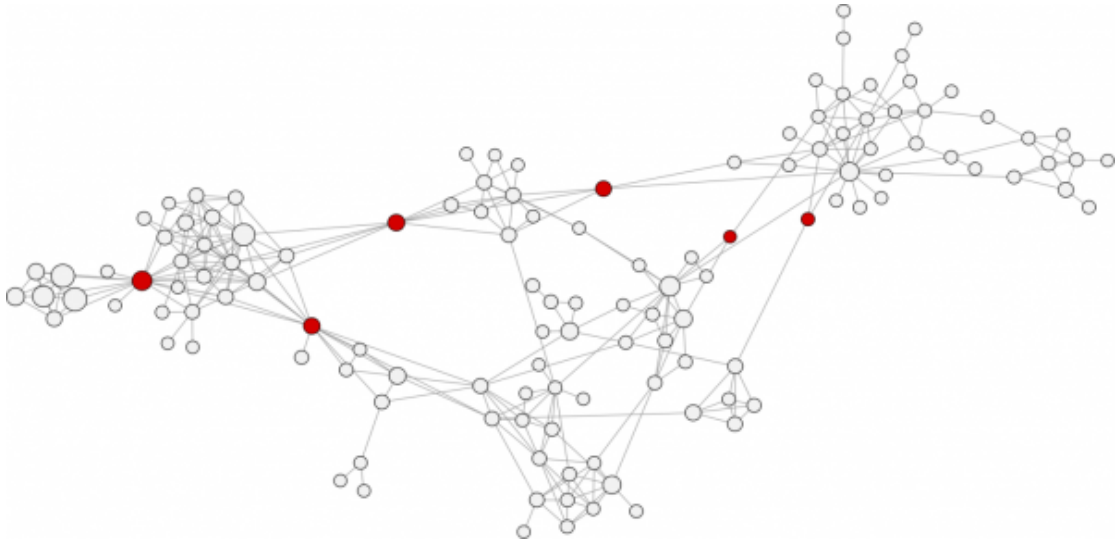



Figure 3-16 from Easley and Kleinberg (2010)

- But, more generally, the nodes that are intuitively the gatekeepers can be determined by **betweenness centrality**.

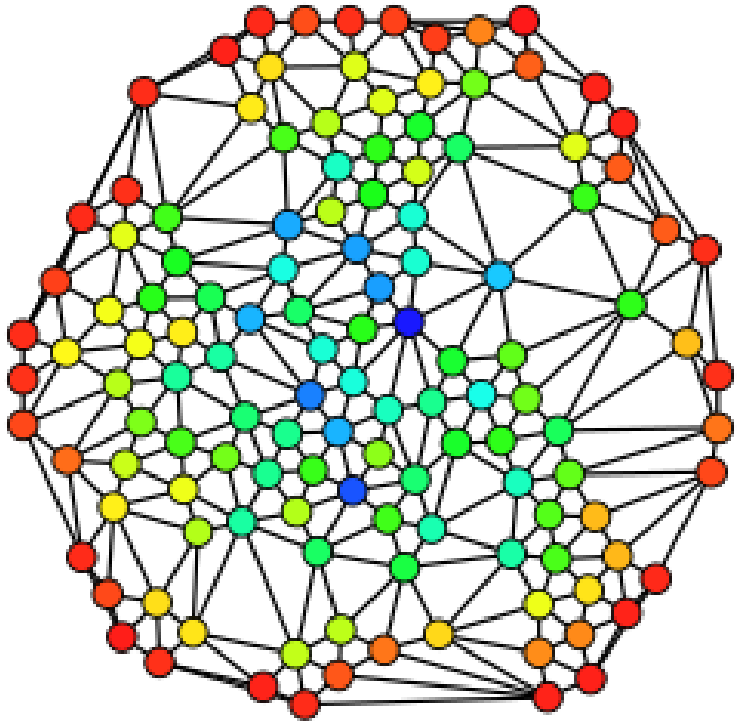
# Betweenness centrality



https://www.linkedin.com/pulse/wtf-do-you-actually-know-who-influencers-walter-pike

- The betweenness centrality of a node V is defined in terms of the proportion of shortest paths that go through V for each pair of nodes.
- Here: the red nodes have high betweenness centrality.
- Note: Easley and Kleinberg talk about ‘flow’: misleading because we only care about shortest paths.

# Betweenness, example



Claudio Rocchini: https://commons.wikimedia.org/wiki/File:Graph_betweenness.svg

- Betweenness: red is minimum; dark blue is maximum.

# Betweenness centrality, formally (from Brandes 2008)

- Directed graph $G =< V, E >$
- $\sigma(s, t)$: number of shortest paths between nodes $s$ and $t$
- $\sigma(s, t|v)$: number of shortest paths between nodes $s$ and $t$ that pass through $v$.
- $C_B(v)$, the betweenness centrality of $v$:

$$C_B(v) = \sum_{s,t \in V} \frac{\sigma(s, t|v)}{\sigma(s, t)}$$

- If $s = t$, then $\sigma(s, t) = 1$
- If $v \in s, t$, then $\sigma(s, t|v) = 0$

# Number of shortest paths

- $\sigma(s,t)$ can be calculated recursively:

$$\sigma(s,t) = \sum_{u \in Pred(t)} \sigma(s,u)$$

  - Pred$(t) = \{u\colon (u,t) \in E, d(s,t) = d(s,u)+1\}$ predecessors of $t$ on shortest path from $s$
  - $d(s,u)$: Distance between nodes $s$ and $u$
- This can be done by running Breadth First search with each node as source $s$ once, for total complexity of $O(V(V+E))$.

# Pairwise dependencies

- There are a cubic number of pairwise dependencies $\delta(s, t|v)$ where:

$$\delta(s, t|v) = \frac{\sigma(s, t|v)}{\sigma(s, t)}$$

- Naive algorithm uses lots of space.
- Brandes (2001) algorithm intuition: the dependencies can be aggregated without calculating them all explicitly.
- Recursive: can calculate dependency of $s$ on $v$ based on dependencies one step further away.

# One-sided dependencies

Define **one-sided dependencies**:

$$\delta(s|v) = \sum_{t \in V} \delta(s, t|v)$$

Then Brandes (2001) shows:

$$\delta(s|v) = \sum_{\substack{(v,w) \in E \\ w:\ d(s,w) = d(s,v)+1}} \frac{\sigma(s,v)}{\sigma(s,w)} . (1 + \delta(s|w))$$

And:

$$C_B(v) = \sum_{s \in V} \delta(s|v)$$

# Brandes algorithm

- Iterate over all vertices s in V
- Calculate $\delta(s|v)$ for all $v \in V$ in two phases:
  1. Breadth-first search, calculating distances and shortest path counts from $s$, push all vertices onto stack as they're visited.
  2. Visit all vertices in reverse order (pop off stack), aggregating dependencies according to equation.

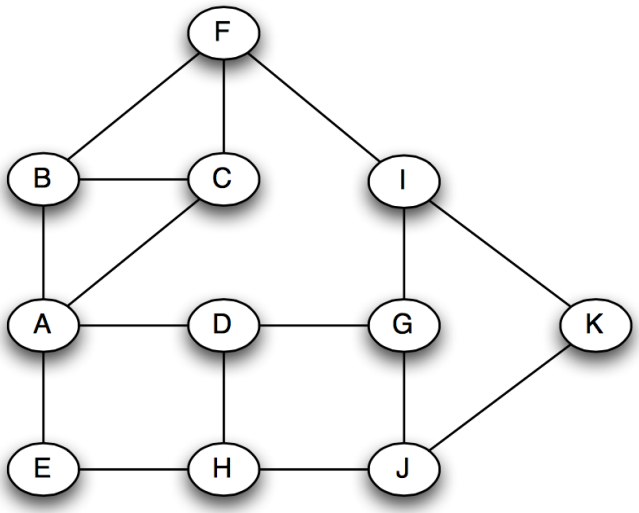
# Brandes (2008) pseudocode

Shortest-path vertex betweenness (Brandes, 2001).

**input**: directed graph $G = (V, E)$
**data**: queue $Q$, stack $S$ (both initially empty)
and for all $v \in V$:
$dist[v]$: distance from source
$Pred[v]$: list of predecessors on shortest paths from source
$\sigma[v]$: number of shortest paths from source to $v \in V$
$\delta[v]$: dependency of source on $v \in V$
**output**: betweenness $c_B[v]$ for all $v \in V$ (initialized to 0)

**for** $s \in V$ **do**
▼ **single-source shortest-paths problem**
▼ **initialization**
**for** $w \in V$ **do** $Pred[w] \leftarrow$ empty list
**for** $t \in V$ **do** $dist[t] \leftarrow \infty$; $\sigma[t] \leftarrow 0$
$dist[s] \leftarrow 0$; $\sigma[s] \leftarrow 1$
enqueue $s \rightarrow Q$
**while** $Q$ *not empty* **do**
dequeue $v \leftarrow Q$; push $v \rightarrow S$
**foreach** *vertex* $w$ *such that* $(v, w) \in E$ **do**
▼ **path discovery** // — $w$ found for the first time?
**if** $dist[w] = \infty$ **then**
$dist[w] \leftarrow dist[v] + 1$
enqueue $w \rightarrow Q$
▼ **path counting** // — edge $(v, w)$ on a shortest path?
**if** $dist[w] = dist[v] + 1$ **then**
$\sigma[w] \leftarrow \sigma[w] + \sigma[v]$
append $v \rightarrow Pred[w]$
▼ **accumulation** // — back-propagation of dependencies
**for** $v \in V$ **do** $\delta[v] \leftarrow 0$
**while** $S$ *not empty* **do**
pop $w \leftarrow S$
**for** $v \in Pred[w]$ **do** $\delta[v] \leftarrow \delta[v] + \frac{\sigma[v]}{\sigma[w]} \cdot (1 + \delta[w])$
**if** $w \neq s$ **then** $c_B[w] \leftarrow c_B[w] + \delta[w]$

# Step 1 - Prepare for BFS tree walk (Node A as $s$)



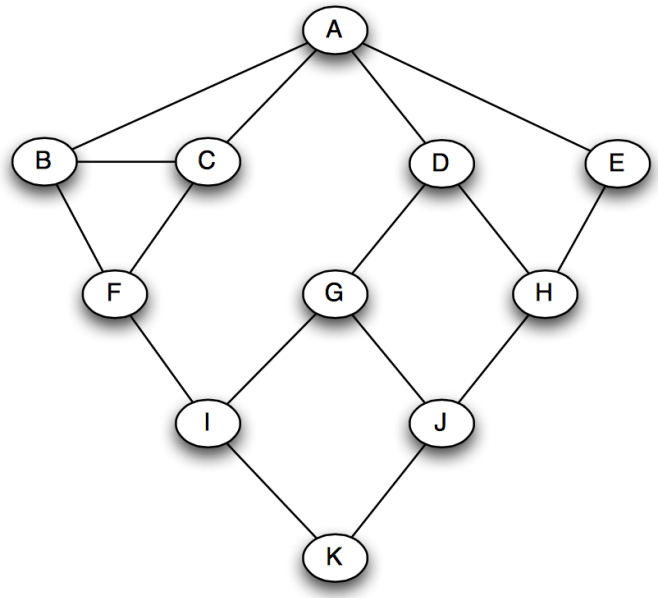
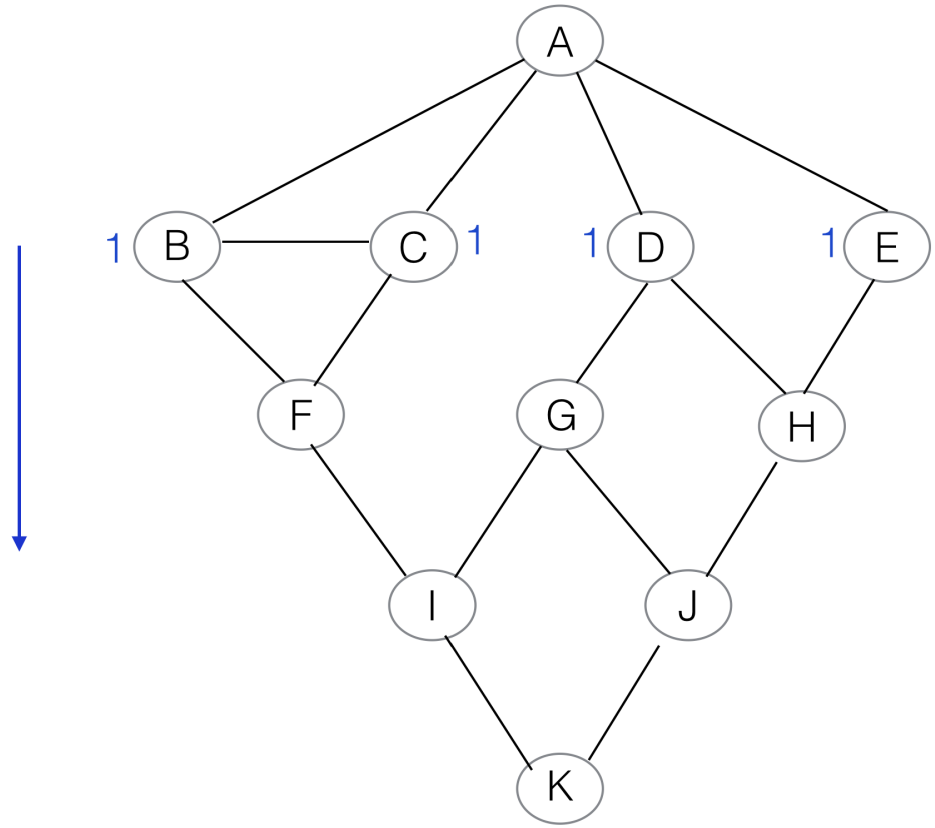




Figure 3-18 from Easley and Kleinberg (2010)

# Brandes (2008) pseudocode: phase 1

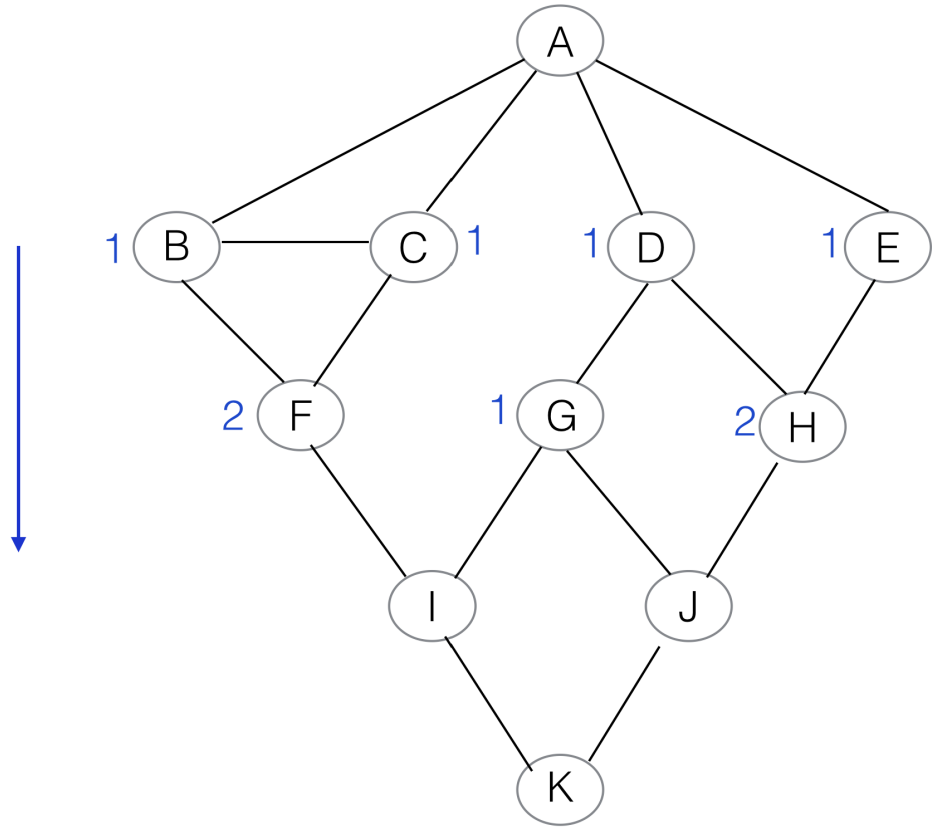
```
while Q not empty do
    dequeue v ← Q;   push v → S
    foreach vertex w such that (v, w) ∈ E do
        ▼ path discovery  // — w found for the first time?
            if dist[w] = ∞ then
                dist[w] ← dist[v] + 1
                enqueue w → Q
        ▼ path counting  // — edge (v, w) on a shortest path?
            if dist[w] = dist[v] + 1 then
                σ[w] ← σ[w] + σ[v]
                append v → Pred[w]
```

# Step 2 - Calculate $\sigma(s, v)$, the number of shortest paths between $s$ and $v$



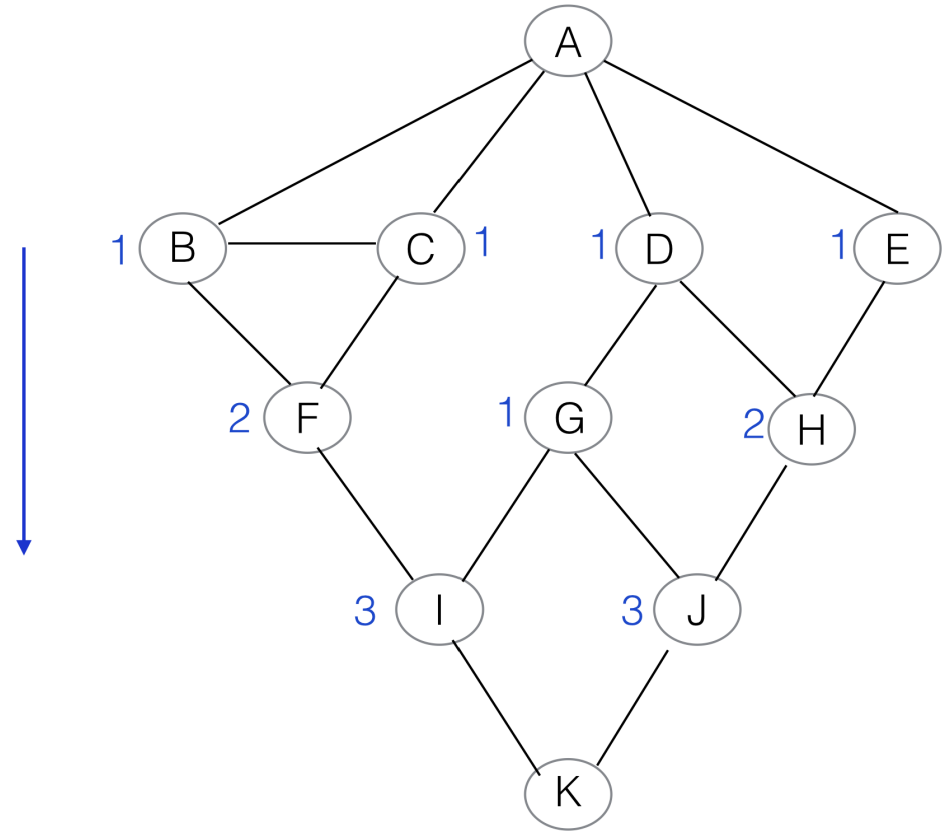

$$\sigma(s,t) = \sum_{u \in Pred(t)} \sigma(s,u)$$

# Step 2 - Calculate $\sigma(s, v)$, the number of shortest paths between $s$ and $v$



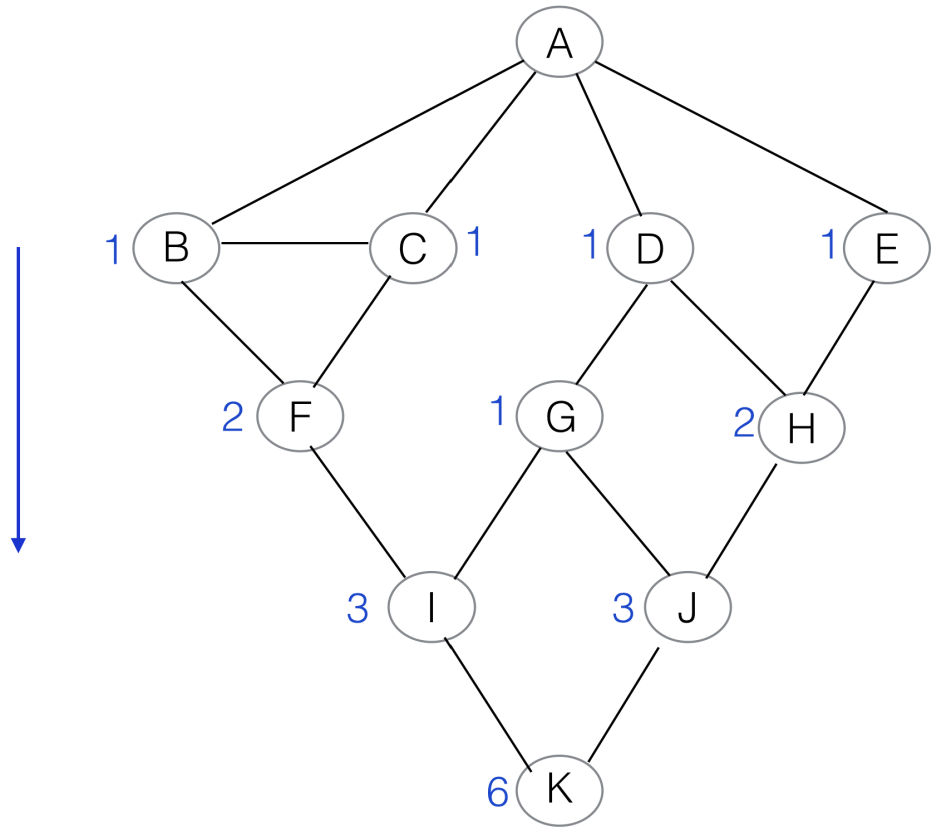

$$\sigma(s,t) = \sum_{u \in Pred(t)} \sigma(s,u)$$

# Step 2 - Calculate $\sigma(s, v)$, the number of shortest paths between $s$ and $v$




$$\sigma(s,t) = \sum_{u \in Pred(t)} \sigma(s,u)$$

# Step 2 - Calculate $\sigma(s, v)$, the number of shortest paths between $s$ and $v$




$$\sigma(s,t) = \sum_{u \in Pred(t)} \sigma(s,u)$$

# Brandes (2008) pseudocode: phase 2

**accumulation** // — back-propagation of dependencies

**for** $v \in V$ **do** $\delta[v] \leftarrow 0$

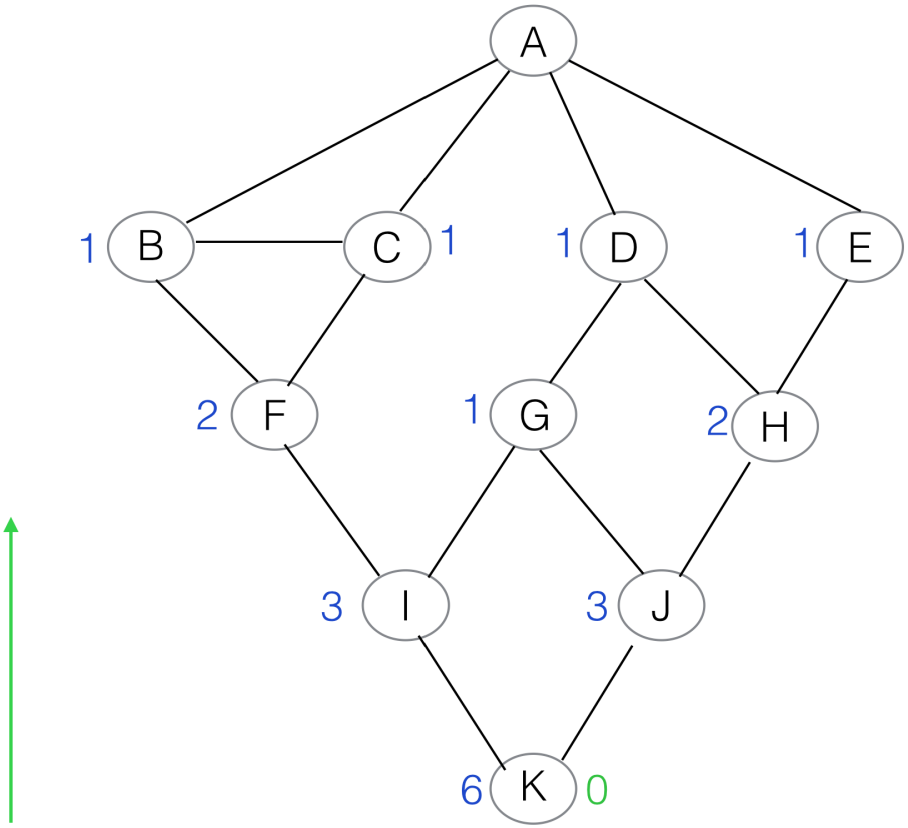
**while** $S$ *not empty* **do**

pop $w \leftarrow S$

**for** $v \in Pred[w]$ **do** $\delta[v] \leftarrow \delta[v] + \frac{\sigma[v]}{\sigma[w]} \cdot (1 + \delta[w])$

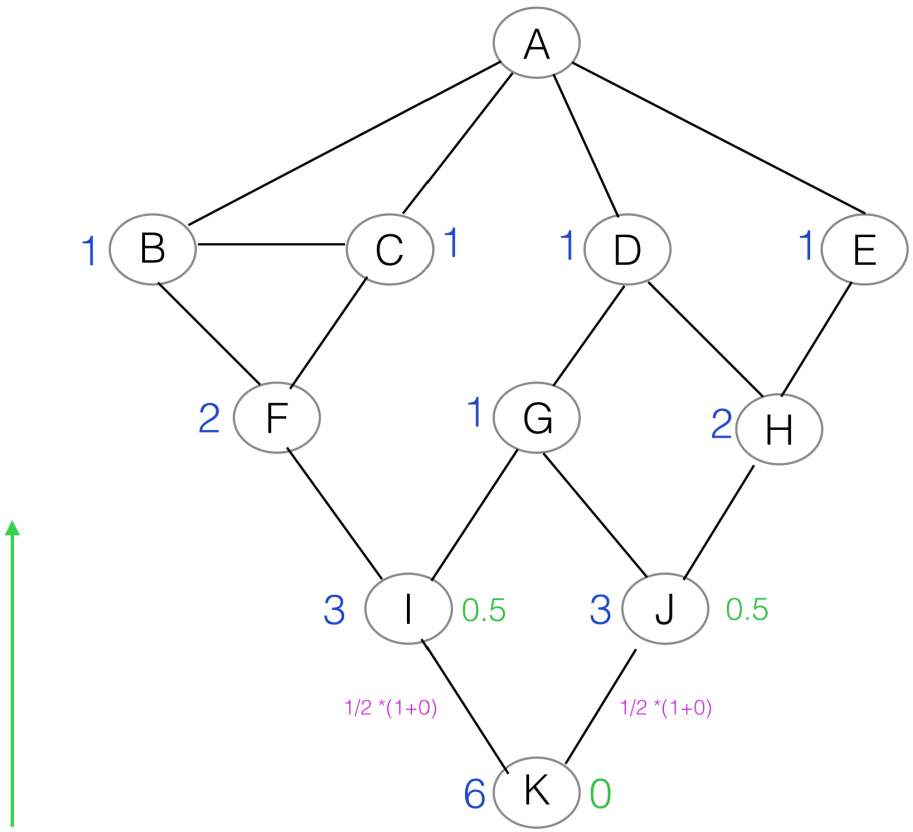
**if** $w \neq s$ **then** $c_B[w] \leftarrow c_B[w] + \delta[w]$

# Step 3 - Calculate $\delta(s|v)$, the dependency of $s$ on $v$



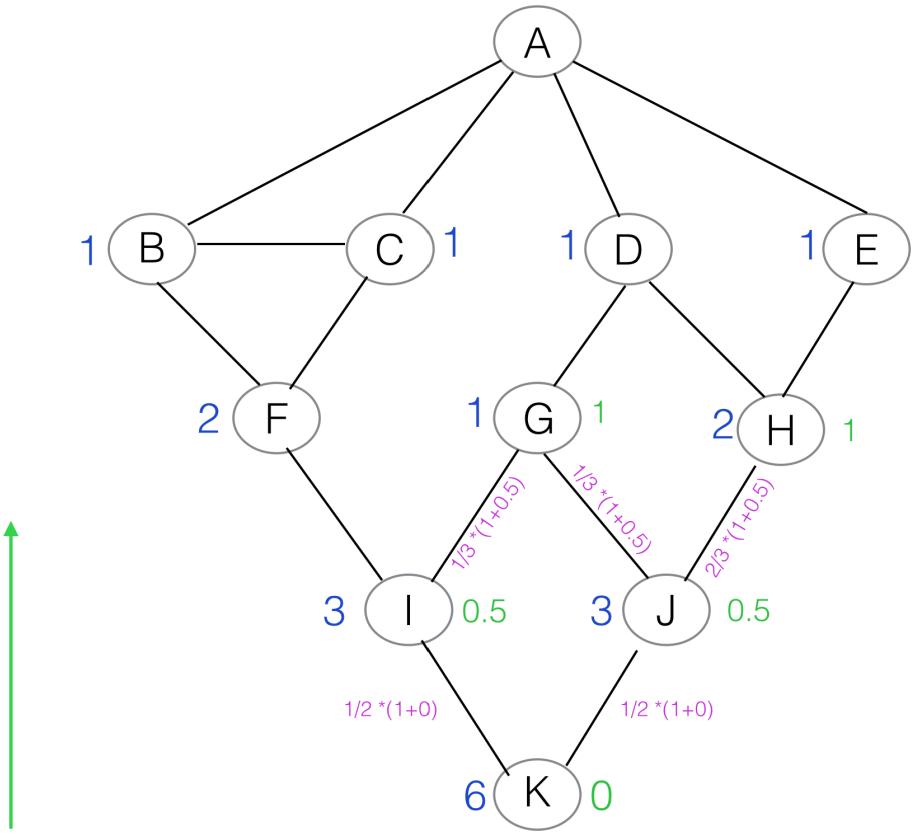

$$\delta(s|v) = \sum_{\substack{(v,w)\in E \\ w:\ d(s,w)=d(s,v)+1}} \sigma(s,v)/\sigma(s,w).(1+\delta(s|w))$$

# Step 3 - Calculate $\delta(s|v)$, the dependency of $s$ on $v$



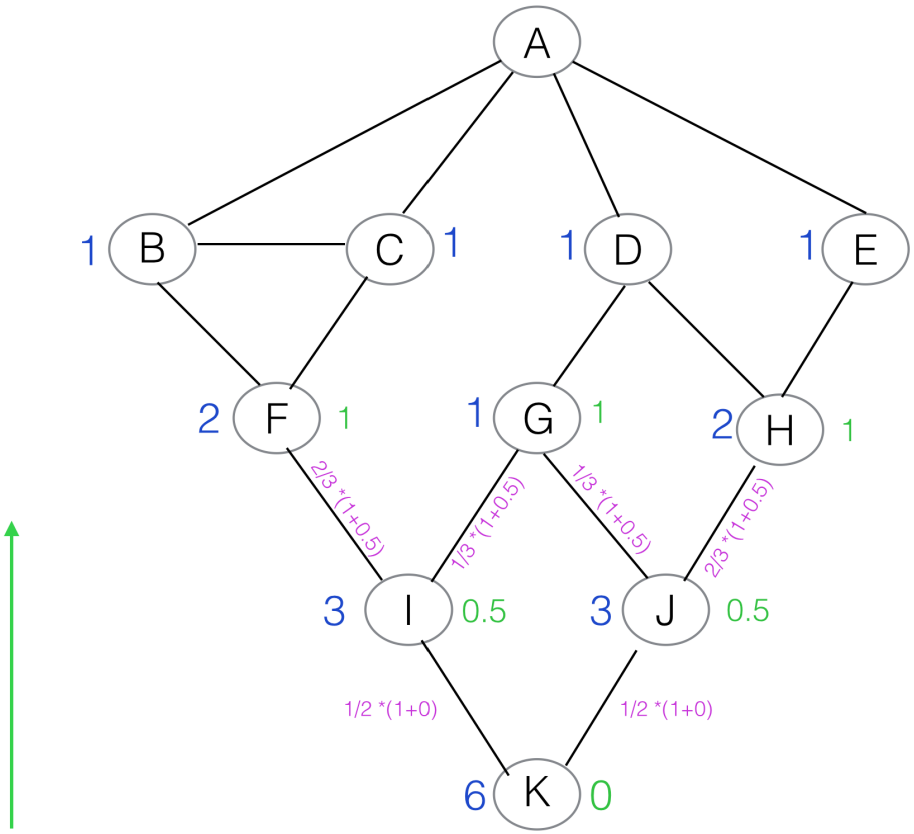

$$\delta(s|v) = \sum_{\substack{(v,w)\in E \\ w:\ d(s,w)=d(s,v)+1}} \sigma(s,v)/\sigma(s,w).(1+\delta(s|w))$$

# Step 3 - Calculate $\delta(s|v)$, the dependency of $s$ on $v$



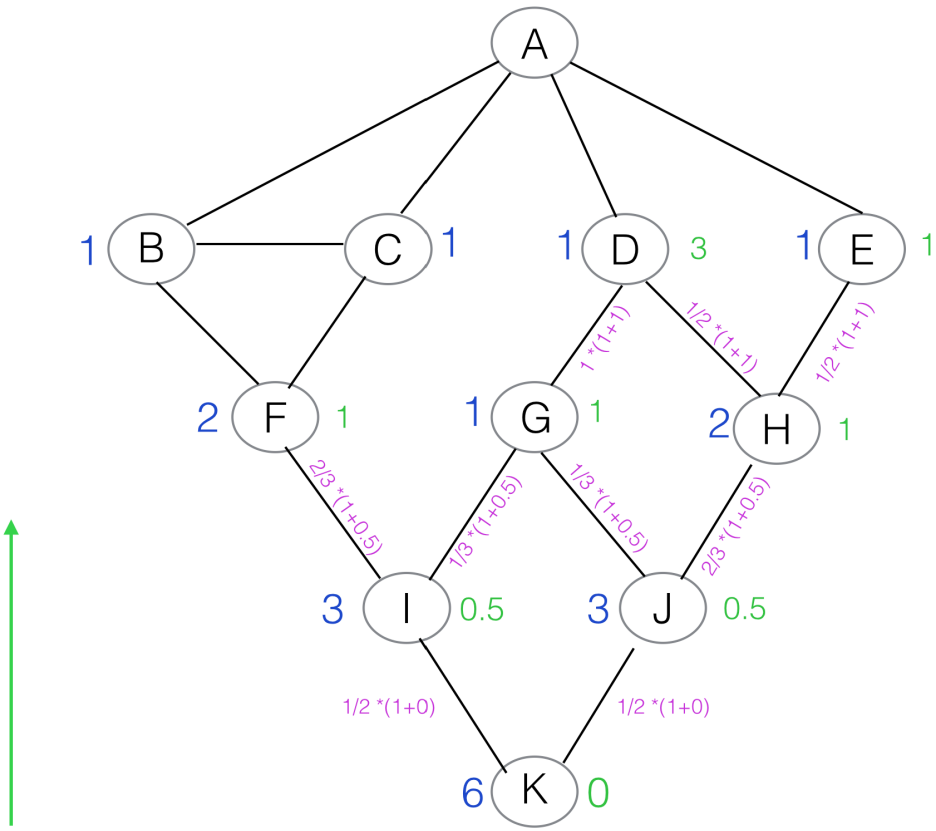

$$\delta(s|v) = \sum_{\substack{(v,w)\in E \\ w:\ d(s,w)=d(s,v)+1}} \sigma(s,v)/\sigma(s,w).(1+\delta(s|w))$$

# Step 3 - Calculate $\delta(s|v)$, the dependency of $s$ on $v$



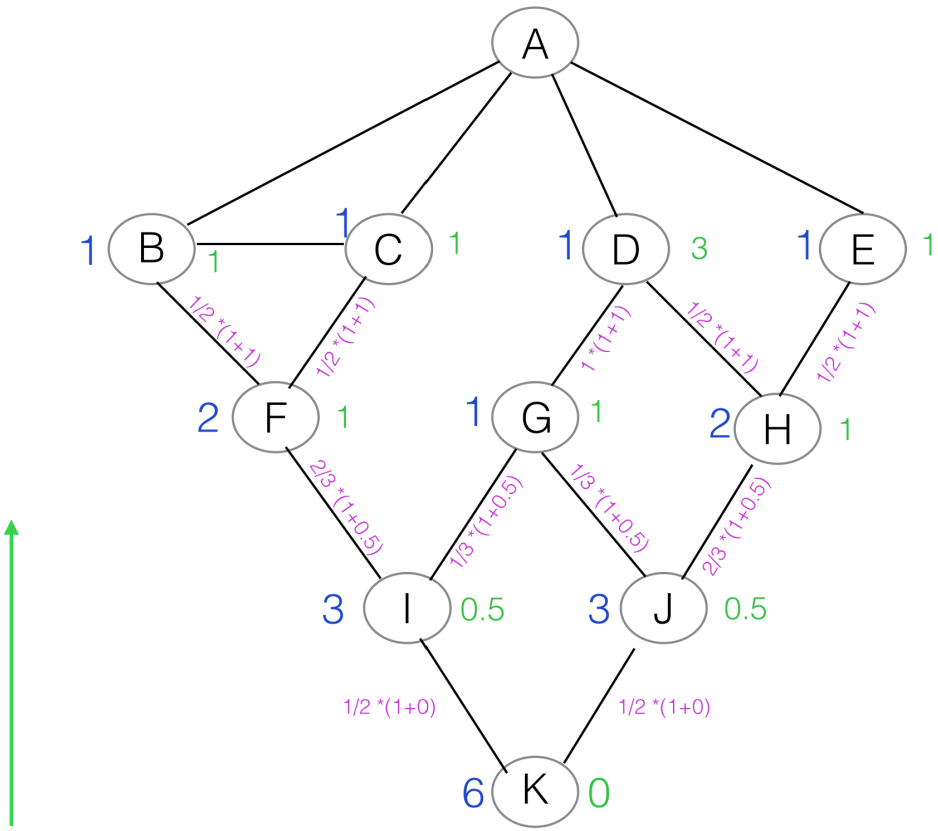

$$\delta(s|v) = \sum_{\substack{(v,w)\in E \\ w:\ d(s,w)=d(s,v)+1}} \sigma(s,v)/\sigma(s,w).(1+\delta(s|w))$$

# Step 3 - Calculate $\delta(s|v)$, the dependency of $s$ on $v$



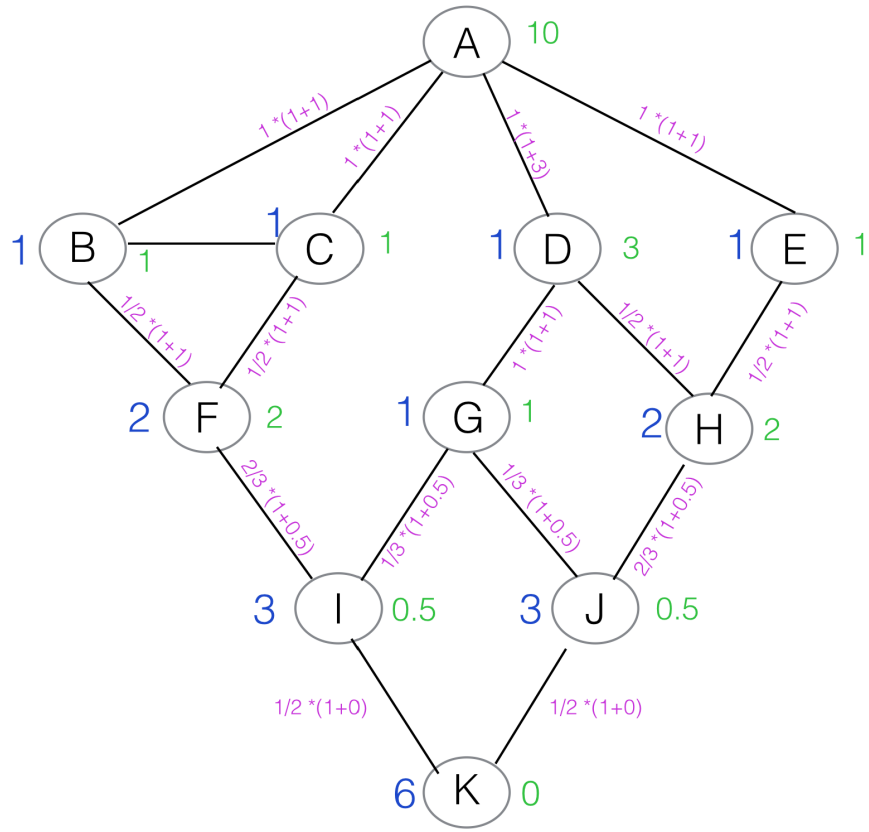

$$\delta(s|v) = \sum_{\substack{(v,w)\in E \\ w:\ d(s,w)=d(s,v)+1}} \sigma(s,v)/\sigma(s,w).(1+\delta(s|w))$$

# Step 3 - Calculate $\delta(s|v)$, the dependency of $s$ on $v$




$$\delta(s|v) = \sum_{\substack{(v,w)\in E \\ w:\ d(s,w)=d(s,v)+1}} \sigma(s,v)/\sigma(s,w).(1+\delta(s|w))$$

# Step 3 - Calculate $\delta(s|v)$, the dependency of $s$ on $v$




$$\delta(s|v) = \sum_{\substack{(v,w)\in E \\ w:\ d(s,w)=d(s,v)+1}} \sigma(s,v)/\sigma(s,w).(1+\delta(s|w))$$

# Step 4 - Calculate betweenness centrality

- You saw one iteration with $s = A$.
- Now perform $V$ iterations, once with each node as source.
- Sum up the $\delta(s|v)$ for each node: this gives the node's betweenness centrality.

# Brandes (2008) pseudocode

Shortest-path vertex betweenness (Brandes, 2001).

**input**: directed graph $G = (V, E)$
**data**: queue $Q$, stack $S$ (both initially empty)
and for all $v \in V$:
$dist[v]$: distance from source
$Pred[v]$: list of predecessors on shortest paths from source
$\sigma[v]$: number of shortest paths from source to $v \in V$
$\delta[v]$: dependency of source on $v \in V$
**output**: betweenness $c_B[v]$ for all $v \in V$ (initialized to 0)

**for** $s \in V$ **do**
▼ **single-source shortest-paths problem**
▼ **initialization**
**for** $w \in V$ **do** $Pred[w] \leftarrow$ empty list
**for** $t \in V$ **do** $dist[t] \leftarrow \infty$; $\sigma[t] \leftarrow 0$
$dist[s] \leftarrow 0$; $\sigma[s] \leftarrow 1$
enqueue $s \rightarrow Q$
**while** $Q$ *not empty* **do**
dequeue $v \leftarrow Q$; push $v \rightarrow S$
**foreach** *vertex* $w$ *such that* $(v, w) \in E$ **do**
▼ **path discovery** // — $w$ found for the first time?
**if** $dist[w] = \infty$ **then**
$dist[w] \leftarrow dist[v] + 1$
enqueue $w \rightarrow Q$
▼ **path counting** // — edge $(v, w)$ on a shortest path?
**if** $dist[w] = dist[v] + 1$ **then**
$\sigma[w] \leftarrow \sigma[w] + \sigma[v]$
append $v \rightarrow Pred[w]$
▼ **accumulation** // — back-propagation of dependencies
**for** $v \in V$ **do** $\delta[v] \leftarrow 0$
**while** $S$ *not empty* **do**
pop $w \leftarrow S$
**for** $v \in Pred[w]$ **do** $\delta[v] \leftarrow \delta[v] + \frac{\sigma[v]}{\sigma[w]} \cdot (1 + \delta[w])$
**if** $w \neq s$ **then** $c_B[w] \leftarrow c_B[w] + \delta[w]$

# Brandes (2008): undirected graphs

- As specified, this is for directed graphs.
- But undirected graphs are easy: the algorithm works in exactly the same way, except that each pair is considered twice, once in each direction.
- Therefore: halve the scores at the end for undirected graphs.
- Brandes (2008) has lots of other variants, including edge betweenness centrality, which we'll use in the next session.

# Today

- Task 11: Implement the Brandes algorithm for efficiently determining the betweenness of each node.

# Literature

- Detailed notes on the Brandes algorithm on course page / Moodle.
- Easley and Kleinberg (2010, page 79-82). But this is an informal description.
- Ulrich Brandes (2001). A faster algorithm for betweenness centrality. *Journal of Mathematical Sociology*. 25:163–177.
- Ulrich Brandes (2008) On variants of shortest-path betweenness centrality and their generic computation. *Social Networks*. 30 (2008), pp. 136–145