L11: Algebraic Path Problems with applications to Internet Routing

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Lectures 4, 5

- Introduction to Combinators for Algebraic Structures (CAS)

Semigroup properties (so far). Call this set of properties $\mathbb{P}_0^{SG}$

$$
\begin{array}{rcl}
\mathbb{AS}(S,\ \bullet) & \equiv & \forall a, b, c \in S,\ a \bullet (b \bullet c) = (a \bullet b) \bullet c \\
\mathbb{IID}(S,\ \bullet,\ \alpha) & \equiv & \forall a \in S,\ a = \alpha \bullet a = a \bullet \alpha \\
\mathbb{ID}(S,\ \bullet) & \equiv & \exists \alpha \in S,\ \mathbb{IID}(S,\ \bullet,\ \alpha) \\
\mathbb{IAN}(S,\ \bullet,\ \omega) & \equiv & \forall a \in S,\ \omega = \omega \bullet a = a \bullet \omega \\
\mathbb{AN}(S,\ \bullet) & \equiv & \exists \omega \in S,\ \mathbb{IAN}(S,\ \bullet,\ \omega) \\
\mathbb{CM}(S,\ \bullet) & \equiv & \forall a, b \in S,\ a \bullet b = b \bullet a \\
\mathbb{SL}(S,\ \bullet) & \equiv & \forall a, b \in S,\ a \bullet b \in \{a,\ b\} \\
\mathbb{IP}(S,\ \bullet) & \equiv & \forall a \in S,\ a \bullet a = a
\end{array}
$$

Bisemigroup properties (so far). Call this set of properties $\mathbb{P}_0^{BS}$

$$
\begin{array}{rcl}
\mathbb{LD}(S,\ \oplus,\ \otimes) & \equiv & \forall a, b, c \in S,\ a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c) \\
\mathbb{RD}(S,\ \oplus,\ \otimes) & \equiv & \forall a, b, c \in S,\ (a \oplus b) \otimes c = (a \otimes c) \oplus (b \otimes c) \\
\mathbb{ZA}(S,\ \oplus,\ \otimes) & \equiv & \exists \overline{0} \in S,\ \mathbb{IID}(S,\ \oplus,\ \overline{0}) \wedge \mathbb{IAN}(S,\ \otimes,\ \overline{0}) \\
\mathbb{OA}(S,\ \oplus,\ \otimes) & \equiv & \exists \overline{1} \in S,\ \mathbb{IID}(S,\ \otimes,\ \overline{1}) \wedge \mathbb{IAN}(S,\ \oplus,\ \overline{1})
\end{array}
$$

Start with an (expandable) set of base structures

Semigroups

S	$\bullet$	α	ω	$\mathbb{CM}$	$\mathbb{SL}$	$\mathbb{IP}$
S	left				$\star$	$\star$
S	right				$\star$	$\star$
$\mathbb{N}$	min		0	$\star$	$\star$	$\star$
$\mathbb{N}$	max	0		$\star$	$\star$	$\star$
$\mathbb{N}$	$+$	0		$\star$		

Bisemigroups

S	$\oplus$	$\otimes$	$\overline{0}$	$\overline{1}$	$\mathbb{LD}$	$\mathbb{RD}$	$\mathbb{ZA}$	$\mathbb{OA}$
$\mathbb{N}$	min	$+$		0	$\star$	$\star$		$\star$
$\mathbb{N}$	max	$+$	0	0	$\star$	$\star$		
$\mathbb{N}$	max	min	0		$\star$	$\star$	$\star$	
$\mathbb{N}$	min	max		0	$\star$	$\star$		$\star$

CAS idea

- We want to develop a set of combinators for constructing new semigroups and bisemigroups.
- A CAS expression will be built from base structures and combinators.
- We want the collection of combinators to be **closed** in the following sense:
  - For each property $\mathbb{Q}$ we can **compute** if a CAS expression satisfies $\mathbb{Q}$ or if it satisfies $\neg\mathbb{Q}$.

Add identity

$$\mathrm{AddId}(\alpha,\ (S,\ \bullet)) \equiv (S \uplus \{\alpha\}, \bullet_{\alpha}^{\mathrm{id}})$$

where $A \uplus B \equiv \{\mathrm{inl}(a) \mid a \in A\} \cup \{\mathrm{inr}(b) \mid b \in B\}$ and

$$a \bullet_{\alpha}^{\mathrm{id}} b \quad \equiv \quad \begin{cases} a & (\text{if } b = \mathrm{inr}(\alpha)) \\ b & (\text{if } a = \mathrm{inr}(\alpha)) \\ \mathrm{inl}(x \bullet y) & (\text{if } a = \mathrm{inl}(x),\ b = \mathrm{inl}(y)) \end{cases}$$

Easy Exercises

$$\begin{array}{rcl} \mathbb{AS}(\mathrm{AddId}(\alpha,\ (S,\ \bullet))) & \Leftrightarrow & \mathbb{AS}(S, \bullet) \\ \mathbb{ID}(\mathrm{AddId}(\alpha,\ (S,\ \bullet))) & \Leftrightarrow & \mathbb{TRUE} \\ \mathbb{AN}(\mathrm{AddId}(\alpha,\ (S,\ \bullet))) & \Leftrightarrow & \mathbb{AN}(S, \bullet) \\ \mathbb{CM}(\mathrm{AddId}(\alpha,\ (S,\ \bullet))) & \Leftrightarrow & \mathbb{CM}(S, \bullet) \\ \mathbb{IP}(\mathrm{AddId}(\alpha,\ (S,\ \bullet))) & \Leftrightarrow & \mathbb{IP}(S, \bullet) \\ \mathbb{SL}(\mathrm{AddId}(\alpha,\ (S,\ \bullet))) & \Leftrightarrow & \mathbb{SL}(S, \bullet) \end{array}$$

Adding an annihilator

$$\mathrm{AddAn}(\omega,\ (S,\ \bullet)) \equiv (S \uplus \{\omega\}, \bullet_{\omega}^{\mathrm{an}})$$

where

$$a \bullet_{\omega}^{\mathrm{an}} b \quad \equiv \quad \begin{cases} \mathrm{inr}(\omega) & (\text{if } b = \mathrm{inr}(\omega)) \\ \mathrm{inr}(\omega) & (\text{if } a = \mathrm{inr}(\omega)) \\ \mathrm{inl}(x \bullet y) & (\text{if } a = \mathrm{inl}(x),\ b = \mathrm{inl}(y)) \end{cases}$$

Easy Exercises

$$\begin{array}{rcl} \mathbb{AS}(\mathrm{AddAn}(\alpha,\ (S,\ \bullet))) & \Leftrightarrow & \mathbb{AS}(S, \bullet) \\ \mathbb{ID}(\mathrm{AddAn}(\alpha,\ (S,\ \bullet))) & \Leftrightarrow & \mathbb{ID}(S, \bullet) \\ \mathbb{AN}(\mathrm{AddAn}(\alpha,\ (S,\ \bullet))) & \Leftrightarrow & \mathbb{TRUE} \\ \mathbb{CM}(\mathrm{AddAn}(\alpha,\ (S,\ \bullet))) & \Leftrightarrow & \mathbb{CM}(S, \bullet) \\ \mathbb{IP}(\mathrm{AddAn}(\alpha,\ (S,\ \bullet))) & \Leftrightarrow & \mathbb{IP}(S, \bullet) \\ \mathbb{SL}(\mathrm{AddAn}(\alpha,\ (S,\ \bullet))) & \Leftrightarrow & \mathbb{SL}(S, \bullet) \end{array}$$

Direct Product of Semigroups

Let $(S,\ \bullet)$ and $(T,\ \diamond)$ be semigroups.

Definition (Direct product semigroup)

The direct product is denoted

$$(S,\ \bullet) \times (T,\ \diamond) \equiv (S \times T,\ \star)$$

where

$$\star = \bullet \times \diamond$$

is defined as

$$(s_1,\ t_1) \star (s_2,\ t_2) = (s_1 \bullet s_2,\ t_1 \diamond t_2).$$

Easy exercises

$$\begin{array}{rcl}
\mathbb{AS}((S,\bullet)\times(T,\diamond)) & \Leftrightarrow & \mathbb{AS}(S,\bullet)\wedge\mathbb{AS}(T,\diamond)\\
\mathbb{ID}((S,\bullet)\times(T,\diamond)) & \Leftrightarrow & \mathbb{ID}(S,\bullet)\wedge\mathbb{ID}(T,\diamond)\\
\mathbb{AN}((S,\bullet)\times(T,\diamond)) & \Leftrightarrow & \mathbb{AN}(S,\bullet)\wedge\mathbb{AN}(T,\diamond)\\
\mathbb{CM}((S,\bullet)\times(T,\diamond)) & \Leftrightarrow & \mathbb{CM}(S,\bullet)\wedge\mathbb{CM}(T,\diamond)\\
\mathbb{IP}((S,\bullet)\times(T,\diamond)) & \Leftrightarrow & \mathbb{IP}(S,\bullet)\wedge\mathbb{IP}(T,\diamond)
\end{array}$$

What about $\mathbb{SL}$?

Consider the product of two selective semigroups, such as $(\mathbb{N},\ \text{min}) \times (\mathbb{N},\ \text{max})$.

$$(10,\ 10) \star (1,\ 3) = (1,\ 10) \notin \{(10,\ 10),\ (1,\ 3)\}$$

The result in this case is not selective!

Direct product and $\mathbb{SL}$?

$$\mathbb{SL}((S,\bullet)\times(T,\diamond)) \ \Leftrightarrow\ (\mathbb{IR}(S,\bullet)\wedge\mathbb{IR}(T,\diamond)) \vee (\mathbb{IL}(S,\bullet)\wedge\mathbb{IL}(T,\diamond))$$

$$\begin{array}{lrcl}
\mathbb{IR} & \text{is right} & \equiv & \forall s,t\in S,\ s\bullet t = t\\
\mathbb{IL} & \text{is left} & \equiv & \forall s,t\in S,\ s\bullet t = s
\end{array}$$

$$\begin{array}{rcl}
\mathbb{IR}((S,\bullet)\times(T,\diamond)) & \Leftrightarrow & \mathbb{IR}(S,\bullet)\wedge\mathbb{IR}(T,\diamond)\\
\mathbb{IL}((S,\bullet)\times(T,\diamond)) & \Leftrightarrow & \mathbb{IL}(S,\bullet)\wedge\mathbb{IL}(T,\diamond)
\end{array}$$

Remember : we have an implicit assumtion that $2 \leqslant |S|$.

Revisit other semigroup constructions ...

To **close** our simple collection $\{\text{AddId}, \text{AddAn}\}$ of semigroup combinators we need

$$\mathbb{P}_1^{SG} \equiv \mathbb{P}_0^{SG} \cup \{\mathbb{IR}, \mathbb{IL}\}$$

and

$$\begin{array}{lcl}
\mathbb{IR}(\text{AddId}(\alpha, (S, \bullet))) & \Leftrightarrow & \mathbb{FALSE} \\
\mathbb{IL}(\text{AddId}(\alpha, (S, \bullet))) & \Leftrightarrow & \mathbb{FALSE} \\
& & \\
\mathbb{IR}(\text{AddAn}(\alpha, (S, \bullet))) & \Leftrightarrow & \mathbb{FALSE} \\
\mathbb{IL}(\text{AddAn}(\alpha, (S, \bullet))) & \Leftrightarrow & \mathbb{FALSE}
\end{array}$$

Operations for adding a zero, a one

$$\begin{array}{lcl}
\text{AddZero}(\overline{0}, (S, \oplus, \otimes)) & \equiv & (S \uplus \{\overline{0}\}, \oplus_{\overline{0}}^{\text{id}}, \otimes_{\overline{0}}^{\text{an}}) \\
& & \\
\text{AddOne}(\overline{1}, (S, \oplus, \otimes)) & \equiv & (S \uplus \{\overline{1}\}, \oplus_{\overline{1}}^{\text{an}}, \otimes_{\overline{1}}^{\text{id}})
\end{array}$$

Easy Exercises

$$\begin{array}{lcl}
\mathbb{LD}(\text{AddZero}(\overline{0}, (S, \oplus, \otimes))) & \Leftrightarrow & \mathbb{LD}(S, \oplus, \otimes) \\
\mathbb{RD}(\text{AddZero}(\overline{0}, (S, \oplus, \otimes))) & \Leftrightarrow & \mathbb{RD}(S, \oplus, \otimes) \\
\mathbb{ZA}(\text{AddZero}(\overline{0}, (S, \oplus, \otimes))) & \Leftrightarrow & \mathbb{TRUE} \\
\mathbb{OA}(\text{AddZero}(\overline{0}, (S, \oplus, \otimes))) & \Leftrightarrow & \mathbb{OA}(S, \oplus, \otimes)
\end{array}$$

Easy Exercises?

Consider left distributivity ($\mathbb{LD}$)

a	b	c	$a \otimes_{\bar{0}}^{\text{an}} (b \oplus_{\bar{0}}^{\text{id}} c)$	$(a \otimes_{\bar{0}}^{\text{an}} b) \oplus_{\bar{0}}^{\text{id}} (a \otimes_{\bar{0}}^{\text{an}} c)$
$\text{inl}(a')$	$\text{inl}(b')$	$\text{inl}(c')$	$\text{inl}(a' \otimes (b' \oplus c'))$	$\text{inl}((a' \otimes b') \oplus (a' \otimes c'))$
$\text{inr}(\bar{0})$	$\text{inl}(b')$	$\text{inl}(c')$	$\text{inr}(\bar{0})$	$\text{inr}(\bar{0})$
$\text{inl}(a')$	$\text{inr}(\bar{0})$	$\text{inl}(c')$	$\text{inl}(a' \oplus c')$	$\text{inl}(a' \oplus c')$
$\text{inl}(a')$	$\text{inl}(b')$	$\text{inr}(\bar{0})$	$\text{inl}(a' \oplus b')$	$\text{inl}(a' \oplus b')$
$\text{inl}(a')$	$\text{inr}(\bar{0})$	$\text{inr}(\bar{0})$	$\text{inr}(\bar{0})$	$\text{inr}(\bar{0})$
$\text{inr}(\bar{0})$	$\text{inr}(\bar{0})$	$\text{inr}(\bar{0})$	$\text{inr}(\bar{0})$	$\text{inr}(\bar{0})$

However, adding a one is more complicated!

Consider left distributivity ($\mathbb{LD}$)

a	b	c	$a \otimes_{\bar{1}}^{\text{id}} (b \oplus_{\bar{1}}^{\text{an}} c)$	$(a \otimes_{\bar{1}}^{\text{id}} b) \oplus_{\bar{1}}^{\text{an}} (a \otimes_{\bar{1}}^{\text{id}} c)$
$\text{inl}(a')$	$\text{inl}(b')$	$\text{inl}(c')$	$\text{inl}(a' \otimes (b' \oplus c'))$	$\text{inl}((a' \otimes b') \oplus (a' \otimes c'))$
$\text{inr}(\bar{1})$	$\text{inl}(b')$	$\text{inl}(c')$	$\text{inl}(b' \oplus c')$	$\text{inl}(b' \oplus c')$
$\text{inl}(a')$	$\text{inr}(\bar{1})$	$\text{inl}(c')$	$\text{inl}(a')$	$\text{inl}((a' \oplus (a' \otimes c'))$
$\text{inl}(a')$	$\text{inl}(b')$	$\text{inr}(\bar{1})$	$\text{inl}(a')$	$\text{inl}((a' \otimes b') \oplus a')$
$\text{inl}(a')$	$\text{inr}(\bar{1})$	$\text{inr}(\bar{1})$	$\text{inl}(a')$	$\text{inl}(a' \oplus a')$
$\text{inr}(\bar{1})$	$\text{inr}(\bar{1})$	$\text{inr}(\bar{1})$	$\text{inr}(\bar{1})$	$\text{inr}(\bar{1})$

Absorption

what does $a = (a \otimes b) \oplus a$ represent?

Let $a \leqslant b \equiv a = a \oplus b$. Then $a = (a \otimes b) \oplus a$ is telling us something else, that

$$a \leqslant a \otimes b.$$

That is, that multiplication is <u>inflationary</u> or <u>non-decreasing</u>.

$\mathbb{AB}$sorption properties (name is from lattice theory)

$$\begin{array}{lcl} \mathbb{RAB}(S,\ \oplus,\ \otimes) & \equiv & \forall a, b \in S,\ a = (a \otimes b) \oplus a = a \oplus (a \otimes b) \\ \mathbb{LAB}(S,\ \oplus,\ \otimes) & \equiv & \forall a, b \in S,\ a = (b \otimes a) \oplus a = a \oplus (b \otimes a) \end{array}$$

To **close** our simple collection $\{\mathrm{AddZero},\ \mathrm{AddOne}\}$ of bisemigroup combinators we need

$$\mathbb{P}_1^{BS} \equiv \mathbb{P}_0^{BS} \cup \{\mathbb{RAB},\ \mathbb{LAB}\}.$$

Rules for absorption for $\mathrm{AddZero}$? Consider $\mathbb{RAB}$

$\mathrm{AddZero}$

a	b	$(a \otimes_{\bar{0}}^{\mathrm{an}} b) \oplus_{\bar{0}}^{\mathrm{id}} a$	$a \oplus_{\bar{0}}^{\mathrm{id}} (a \otimes_{\bar{0}}^{\mathrm{an}} b)$
$\mathrm{inl}(a')$	$\mathrm{inl}(b')$	$\mathrm{inl}((a' \otimes b') \oplus a)$	$\mathrm{inl}(a' \oplus (a' \otimes b'))$
$\mathrm{inr}(\bar{0})$	$\mathrm{inl}(b')$	$\mathrm{inr}(\bar{0})$	$\mathrm{inr}(\bar{0})$
$\mathrm{inl}(a')$	$\mathrm{inr}(\bar{0})$	$\mathrm{inl}(a')$	$\mathrm{inl}(a')$
$\mathrm{inr}(\bar{0})$	$\mathrm{inr}(\bar{0})$	$\mathrm{inr}(\bar{0})$	$\mathrm{inr}(\bar{0})$

$$\begin{array}{lcl} \mathbb{RAB}(\mathrm{AddZero}(\bar{0},\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & \mathbb{RAB}(S,\ \oplus,\ \otimes) \\ \mathbb{LAB}(\mathrm{AddZero}(\bar{0},\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & \mathbb{LAB}(S,\ \oplus,\ \otimes) \end{array}$$

Rules for absorption for AddOne? Consider $\mathbb{RAB}$

AddOne

a	b	$(a \otimes_{\overline{1}}^{\mathrm{id}} b) \oplus_{\overline{1}}^{\mathrm{an}} a$	$a \oplus_{\overline{1}}^{\mathrm{an}} (a \otimes_{\overline{1}}^{\mathrm{id}} b)$
$\mathrm{inl}(a')$	$\mathrm{inl}(b')$	$\mathrm{inl}((a' \otimes b') \oplus a)$	$\mathrm{inl}(a' \oplus (a' \otimes b'))$
$\mathrm{inr}(\overline{1})$	$\mathrm{inl}(b')$	$\mathrm{inr}(\overline{1})$	$\mathrm{inr}(\overline{1})$
$\mathrm{inl}(a')$	$\mathrm{inr}(\overline{1})$	$\mathrm{inl}(a')$	$\mathrm{inl}(a' \oplus a')$
$\mathrm{inr}(\overline{1})$	$\mathrm{inr}(\overline{1})$	$\mathrm{inr}(\overline{1})$	$\mathrm{inr}(\overline{1})$

Property management for AddOne

$$
\begin{array}{rcl}
\mathbb{LD}(\mathrm{AddOne}(\overline{1},\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & \mathbb{LD}(S,\ \oplus,\ \otimes) \wedge \mathbb{RAB}(S,\ \oplus,\ \otimes) \\
 & & \wedge\ \mathbb{IP}(S,\ \oplus) \\
\mathbb{RD}(\mathrm{AddOne}(\overline{1},\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & \mathbb{RD}(S,\ \oplus,\ \otimes) \wedge \mathbb{LAB}(S,\ \oplus,\ \otimes) \\
 & & \wedge\ \mathbb{IP}(S,\ \oplus) \\
\mathbb{ZA}(\mathrm{AddOne}(\overline{1},\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & \mathbb{ZA}(S,\ \oplus,\ \otimes) \\
\mathbb{OA}(\mathrm{AddOne}(\overline{1},\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & \mathbb{TRUE} \\
\mathbb{RAB}(\mathrm{AddOne}(\overline{1},\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & \mathbb{RAB}(S,\ \oplus,\ \otimes) \wedge \mathbb{IP}(S,\ \oplus) \\
\mathbb{LAB}(\mathrm{AddOne}(\overline{1},\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & \mathbb{LAB}(S,\ \oplus,\ \otimes) \wedge \mathbb{IP}(S,\ \oplus)
\end{array}
$$

Lexicographic Product of Semigroups

Lexicographic product semigroup

Suppose that semigroup $(S,\ \bullet)$ is commutative, idempotent, and selective and that $(T,\ \diamond)$ is a semigroup.

$$(S,\ \bullet)\ \vec{\times}\ (T,\ \diamond) \equiv (S \times T,\ \star)$$

where $\star \equiv \bullet\ \vec{\times} \diamond$ is defined as

$$(s_1,\ t_1) \star (s_2,\ t_2) = \begin{cases} (s_1 \bullet s_2,\ t_1 \diamond t_2) & s_1 = s_1 \bullet s_2 = s_2 \\ (s_1 \bullet s_2,\ t_1) & s_1 = s_1 \bullet s_2 \neq s_2 \\ (s_1 \bullet s_2,\ t_2) & s_1 \neq s_1 \bullet s_2 = s_2 \end{cases}$$

Examples

$(\mathbb{N},\ \min)\ \vec{\times}\ (\mathbb{N},\ \min)$

$$\begin{array}{rcl} (1,\ 17) \star (2, 3) & = & (1, 17) \\ (2,\ 17) \star (2, 3) & = & (2, 3) \\ (2,\ 3) \star (2, 3) & = & (2, 3) \end{array}$$

$(\mathbb{N},\ \min)\ \vec{\times}\ (\mathbb{N},\ \max)$

$$\begin{array}{rcl} (1,\ 17) \star (2, 3) & = & (1, 17) \\ (2,\ 17) \star (2, 3) & = & (2, 17) \\ (2,\ 3) \star (2, 3) & = & (2, 3) \end{array}$$

$(\mathbb{N},\ \max)\ \vec{\times}\ (\mathbb{N},\ \min)$

$$\begin{array}{rcl} (1,\ 17) \star (2, 3) & = & (2, 3) \\ (2,\ 17) \star (2, 3) & = & (2, 3) \\ (2,\ 3) \star (2, 3) & = & (2, 3) \end{array}$$

Assuming $\mathbb{CM}(S, \bullet) \wedge \mathbb{SL}(S, \bullet)$

$$\begin{array}{rcl}
\mathbb{AS}((S, \bullet)\vec{\times}(T, \diamond)) & \Leftrightarrow & \mathbb{AS}(S, \bullet) \wedge \mathbb{AS}(T, \diamond) \\
\mathbb{ID}((S, \bullet)\vec{\times}(T, \diamond)) & \Leftrightarrow & \mathbb{ID}(S, \bullet) \wedge \mathbb{ID}(T, \diamond) \\
\mathbb{AN}((S, \bullet)\vec{\times}(T, \diamond)) & \Leftrightarrow & \mathbb{AN}(S, \bullet) \wedge \mathbb{AN}(T, \diamond) \\
\mathbb{CM}((S, \bullet)\vec{\times}(T, \diamond)) & \Leftrightarrow & \mathbb{CM}(T, \diamond) \\
\mathbb{IP}((S, \bullet)\vec{\times}(T, \diamond)) & \Leftrightarrow & \mathbb{IP}(T, \diamond) \\
\mathbb{SL}((S, \bullet)\vec{\times}(T, \diamond)) & \Leftrightarrow & \mathbb{SL}(T, \diamond) \\
\mathbb{IR}((S, \bullet)\vec{\times}(T, \diamond)) & \Leftrightarrow & \mathbb{FALSE} \\
\mathbb{IL}((S, \bullet)\vec{\times}(T, \diamond)) & \Leftrightarrow & \mathbb{FALSE}
\end{array}$$

All easy, except for $\mathbb{AS}$ (very tedious!). We are assuming commutativity and selectivity in order to guarantee associativity.

Lexicographic product for Bi-Semigroups

Assume $\mathbb{AS}(S, \oplus_S) \wedge \mathbb{AS}(T, \oplus_T) \wedge \mathbb{CM}(S, \oplus_S) \wedge \mathbb{SL}(S, \oplus_S)$

Let

$$(S, \oplus_S, \otimes_S) \vec{\times} (T, \oplus_T, \otimes_T) \equiv (S \times T, \oplus_S \vec{\times} \oplus_T, \otimes_S \times \otimes_T)$$

That is, the additive component is a lexicographic product, and the multiplicative component is a direct product.

Examples

$\oplus = \min \vec{\times} \max$, $\otimes = + \times \min$

$$
\begin{aligned}
(3,10) \otimes ((17,21) \oplus (11,4)) &= (3,10) \otimes (11,4) \\
&= (14,4) \\
((3,10) \otimes (17,21)) \oplus ((3,10) \otimes (11,4)) &= (20,10) \oplus (14,4) \\
&= (14,4)
\end{aligned}
$$

$\oplus = \max \vec{\times} \min$, $\otimes = \min \times +$

$$
\begin{aligned}
(3,10) \otimes ((17,21) \oplus (11,4)) &= (3,10) \otimes (17,21) \\
&= (3,31) \\
((3,10) \otimes (17,21)) \oplus ((3,10) \otimes (11,4)) &= (3,31) \oplus (3,14) \\
&= (3,14)
\end{aligned}
$$

Distributivity?

Theorem: If $\oplus_S$ is commutative and selective, then

$$
\begin{aligned}
&\mathbb{LD}((S,\ \oplus_S,\ \otimes_S)\ \vec{\times}\ (T,\ \oplus_T,\ \otimes_T)) \Leftrightarrow \\
&\quad \mathbb{LD}(S,\ \oplus_S,\ \otimes_S) \wedge \mathbb{LD}(T,\ \oplus_T,\ \otimes_T) \wedge (\mathbb{LC}(S,\ \otimes_S) \vee \mathbb{LK}(T,\ \otimes_T))
\end{aligned}
$$

$$
\begin{aligned}
&\mathbb{RD}((S,\ \oplus_S,\ \otimes_S)\ \vec{\times}\ (T,\ \oplus_T,\ \otimes_T)) \Leftrightarrow \\
&\quad \mathbb{RD}(S,\ \oplus_S,\ \otimes_S) \wedge \mathbb{RD}(T,\ \oplus_T,\ \otimes_T) \wedge (\mathbb{RC}(S,\ \otimes_S) \vee \mathbb{RK}(T,\ \otimes_T))
\end{aligned}
$$

Left and Right Cancellative

$$
\begin{aligned}
\mathbb{LC}(X,\ \bullet) &\equiv \forall a, b, c \in X,\ c \bullet a = c \bullet b \Rightarrow a = b \\
\mathbb{RC}(X,\ \bullet) &\equiv \forall a, b, c \in X,\ a \bullet c = b \bullet c \Rightarrow a = b
\end{aligned}
$$

Left and Right Constant

$$
\begin{aligned}
\mathbb{LK}(X,\ \bullet) &\equiv \forall a, b, c \in X,\ c \bullet a = c \bullet b \\
\mathbb{RK}(X,\ \bullet) &\equiv \forall a, b, c \in X,\ a \bullet c = b \bullet c
\end{aligned}
$$

Why bisemigroups?

But wait! How could any semiring satisfy either of these properties?

$$\begin{array}{rcl} \mathbb{LC}(X,\ \bullet) & \equiv & \forall a,b,c \in X,\ c \bullet a = c \bullet b \Rightarrow a = b \\ \mathbb{LK}(X,\ \bullet) & \equiv & \forall a,b,c \in X,\ c \bullet a = c \bullet b \end{array}$$

- For $\mathbb{LC}$, note that we always have $\overline{0} \otimes a = \overline{0} \otimes b$, so $\mathbb{LC}$ could only hold when $S = \{\overline{0}\}$.
- For $\mathbb{LK}$, let $a = \overline{1}$ and $b = \overline{0}$ and $\mathbb{LK}$ leads to the conclusion that every c is equal to $\overline{0}$ (again!).

Normally we will add a zero and/or a one as the last step(s) of constructing a semiring. Alternatively, we might want to complicate our properties so that things work for semirings. A design trade-off!

Proof of $\Leftarrow$ for $\mathbb{LD}$ (Very carefully ...)

Assume

$$\begin{array}{ll} (1) & \mathbb{LD}(S,\ \oplus_S,\ \otimes_S) \\ (2) & \mathbb{LD}(T,\ \oplus_T,\ \otimes_T) \\ (3) & \mathbb{LC}(S,\ \otimes_S) \vee \mathbb{LK}(T,\ \otimes_T) \\ (4) & \mathbb{IP}(S,\ \oplus_S). \end{array}$$

Let $\oplus \equiv \oplus_S \vec{\times} \oplus_T$ and $\otimes \equiv \otimes_S \times \otimes_T$. Suppose

$$(s_1, t_1),\ (s_2, t_2),\ (s_3, t_3) \in S \times T.$$

We want to show that

$$\begin{array}{rcl} \text{lhs} & \equiv & (s_1,\ t_1) \otimes ((s_2,\ t_2) \oplus (s_3,\ t_3)) \\ & = & ((s_1,\ t_1) \otimes (s_2,\ t_2)) \oplus ((s_1,\ t_1) \otimes (s_3,\ t_3)) \\ & \equiv & \text{rhs} \end{array}$$

Proof of $\Leftarrow$ for $\mathbb{LD}$

We have

$$
\begin{array}{rcl}
\mathrm{lhs} & \equiv & (s_1,\ t_1) \otimes ((s_2,\ t_2) \oplus (s_3,\ t_3)) \\
 & = & (s_1,\ t_1) \otimes (s_2 \oplus_S s_3,\ t_{\mathrm{lhs}}) \\
 & = & (s_1 \otimes_S (s_2 \oplus_S s_3),\ t_1 \otimes_T t_{\mathrm{lhs}}) \\
 & & \\
\mathrm{rhs} & \equiv & ((s_1,\ t_1) \otimes (s_2,\ t_2)) \oplus ((s_1,\ t_1) \otimes (s_3,\ t_3)) \\
 & = & (s_1 \otimes_S s_2,\ t_1 \otimes_T t_2) \oplus (s_1 \otimes_S s_3,\ t_1 \otimes_T t_3) \\
 & = & ((s_1 \otimes_S s_2) \oplus_S (s_1 \otimes_S s_3),\ t_{\mathrm{rhs}}) \\
 & =_{(1)} & (s_1 \otimes_S (s_2 \oplus_S s_3),\ t_{\mathrm{rhs}})
\end{array}
$$

where t_{lhs} and t_{rhs} are determined by the appropriate case in the definition of $\oplus$. Finally, note that

$$\mathrm{lhs} = \mathrm{rhs} \Leftrightarrow t_{\mathrm{rhs}} = t_1 \otimes t_{\mathrm{lhs}}.$$

Proof by cases on $s_2 \oplus_S s_3$

Case 1 : $s_2 = s_2 \oplus_S s_3 = s_3$. Then $t_{\mathrm{lhs}} = t_2 \oplus_T t_3$ and

$$t_1 \otimes_T t_{\mathrm{lhs}} = t_1 \otimes_T (t_2 \oplus_T t_3) =_{(2)} (t_1 \otimes_T t_2) \oplus_T (t_1 \otimes_T t_3).$$

Since $s_2 = s_3$ we have $s_1 \otimes_S s_2 = s_1 \otimes_S s_3$ and

$$s_1 \otimes_S s_2 =_{(4)} (s_1 \otimes_S s_2) \oplus_S (s_1 \otimes_S s_3) =_{(4)} s_1 \otimes_S s_3.$$

Therefore,

$$t_{\mathrm{rhs}} = (t_1 \otimes_T t_2) \oplus (t_1 \otimes_T t_3) = t_1 \otimes_T t_{\mathrm{lhs}}.$$

Case 2 : $s_2 = s_2 \oplus_S s_3 \neq s_3$. Then $t_{\mathrm{lhs}} = t_2$ and

$$t_1 \otimes_T t_{\mathrm{lhs}} = t_1 \otimes_T t_2.$$

Since $s_2 = s_2 \oplus_S s_3$ we have

$$s_1 \otimes_S s_2 = s_1 \otimes_S (s_2 \oplus_S s_3) =_{(1)} (s_1 \otimes_S s_2) \oplus_S (s_1 \otimes_S s_3).$$

Case 2.1 $s_1 \otimes_S s_2 \neq s_1 \otimes_S s_3$. Then $t_{\mathrm{rhs}} = t_1 \otimes_T t_2 = t_1 \otimes_T t_{\mathrm{lhs}}$.
Case 2.2 $s_1 \otimes_S s_2 = s_1 \otimes_S s_3$. Then

$$t_{\mathrm{rhs}} = (t_1 \otimes_T t_2) \oplus_T (t_1 \otimes_T t_3) =_{(2)} t_1 \otimes_T (t_2 \oplus_T t_3)$$

We need to consider two subcases.
Case 2.2.1: Assume $\mathbb{LC}(S, \otimes_S)$. But $s_1 \otimes_S s_2 = s_1 \otimes_S s_3 \Rightarrow s_2 = s_3$, which is a contradiction.
Case 2.2.2 : Assume $\mathbb{LK}(T, \otimes_T)$. In this case we know

$$\forall a, b \in X, \ t_1 \otimes_T a = t_1 \otimes_T b.$$

Letting $a = t_2 \oplus_T t_3$ and $b = t_2$ we have

$$t_{\mathrm{rhs}} = t_1 \otimes_T (t_2 \oplus_T t_3) = t_1 \otimes_T t_2 = t_1 \otimes_T t_{\mathrm{lhs}}.$$

Case 3 : $s_2 \neq s_2 \oplus_S s_3 = s_3$. Similar to Case 2.

Other direction, $\Rightarrow$ (Very carefully ...)

Prove this:

$$\neg\mathbb{LD}(S, \oplus_S, \otimes_S) \vee \neg\mathbb{LD}(T, \oplus_T, \otimes_T) \vee (\neg\mathbb{LC}(S, \otimes_S) \wedge \neg\mathbb{LK}(T, \otimes_T))$$
$$\Rightarrow \ \neg\mathbb{LD}((S, \oplus_S, \otimes_S) \vec{\times} (T, \oplus_T, \otimes_T)).$$

Case 1: $\neg\mathbb{LD}(S, \oplus_S, \otimes_S)$. That is

$$\exists a, b, c \in S, \ a \otimes_S (b \oplus_S c) \neq (a \otimes_S b) \oplus_S (a \otimes_S c).$$

Pick any $t \in T$. Then for some $t_1, t_2, t_3 \in T$ we have

$$\begin{array}{ll} & (a, \ t) \otimes ((b, \ t) \oplus (c, \ t)) \\ = & (a, \ t) \otimes (b \oplus_S c, \ t_1) \\ = & (a, \otimes_S (b \oplus_S c), \ t_2) \\ \neq & ((a \otimes_S b) \oplus_S (a \otimes_S c), \ t_3) \\ = & (a \otimes_S b, \ t \otimes_T t) \oplus (a \otimes_S c, \ t \otimes_T t) \\ = & ((a, \ t) \otimes (b, \ t)) \oplus ((a, \ t) \otimes (c, \ t)) \end{array}$$

Case 2: $\neg\mathbb{LD}(T, \oplus_T, \otimes_T)$. Similar.

Case 3: $(\neg\mathbb{LC}(S,\ \otimes_S) \wedge \neg\mathbb{LK}(T,\ \otimes_T))$. That is

$$\exists a, b, c \in S,\ c \otimes_S a = c \otimes_S b \wedge a \neq b$$

and

$$\exists x, y, z \in T,\ z \otimes_T x \neq z \otimes_T y.$$

Since $\oplus_S$ is selective and $a \neq b$, we have $a = a \oplus_S b$ or $b = a \oplus_S b$.
Case 3.1 : Assume $a = a \oplus_S b \neq b$.
Suppose that $t_1, t_2, t_3 \in T$. Then

$$\begin{array}{lcl} \text{lhs} & \equiv & (c,\ t_1) \otimes ((a,\ t_2) \oplus (b,\ t_3)) \\ & = & (c,\ t_1) \otimes (a,\ t_2) \\ & = & (c \otimes_S a,\ t_1 \otimes_T t_2) \end{array}$$

$$\begin{array}{lcl} \text{rhs} & \equiv & ((c,\ t_1) \otimes (a,\ t_2)) \oplus ((c,\ t_1) \otimes (b,\ t_3)) \\ & = & (c \otimes_S a,\ t_1 \otimes_T t_2) \oplus (c \otimes_S b,\ t_1 \otimes_T t_3) \\ & = & (c \otimes_S a,\ (t_1 \otimes_T t_2) \oplus_T (t_1 \otimes_T t_3)) \end{array}$$

Our job now is to select t_1, t_2, t_3 so that

$$t_{\text{lhs}} \equiv t_1 \otimes_T t_2 \neq (t_1 \otimes_T t_2) \oplus_T (t_1 \otimes_T t_3) \equiv t_{\text{rhs}}.$$

We don't have very much to work with! Only

$$\exists x, y, z \in T,\ z \otimes_T x \neq z \otimes_T y.$$

In addition, we can assume $\mathbb{LD}(T,\ \oplus_T,\ \otimes_T)$ (otherwise, use Case 2!), so

$$t_{\text{rhs}} = t_1 \otimes_T (t_2 \oplus_T t_3).$$

We need to select t_1, t_2, t_3 so that

$$t_{\text{lhs}} \equiv t_1 \otimes_T t_2 \neq t_1 \otimes_T (t_2 \oplus_T t_3) \equiv t_{\text{rhs}}.$$

Case 3.1.1: $z \otimes_T x = z \otimes_T (x \oplus_T y)$. Then letting $t_1 = z$, $t_2 = y$, and $t_3 = x$ we have

$$t_{\text{lhs}} = z \otimes_T y \neq z \otimes_T x = z \otimes_T (x \oplus_T y) = z \otimes_T (y \oplus_T x) = t_{\text{rhs}}.$$

Case 3.1.2: $z \otimes_T x \neq z \otimes_T (x \oplus_T y)$. Then letting $t_1 = z$, $t_2 = x$, and $t_3 = y$ we have

$$t_{\text{lhs}} = z \otimes_T x \neq z \otimes_T (x \oplus_T y) = t_{\text{rhs}}.$$

Case 3.2 : Assume $b = a \oplus_S b \neq a$.
Suppose that $t_1, t_2, t_3 \in T$. Then

$$\begin{array}{lcl}
\text{lhs} & \equiv & (c,\ t_1) \otimes ((a,\ t_2) \oplus (b,\ t_3)) \\
& = & (c,\ t_1) \otimes (b,\ t_3) \\
& = & (c \otimes_S b,\ t_1 \otimes_T t_3)
\end{array}$$

$$\begin{array}{lcl}
\text{rhs} & \equiv & ((c,\ t_1) \otimes (a,\ t_2)) \oplus ((c,\ t_1) \otimes (b,\ t_3)) \\
& = & (c \otimes_S a,\ t_1 \otimes_T t_2) \oplus (c \otimes_S b,\ t_1 \otimes_T t_3) \\
& = & (c \otimes_S b,\ (t_1 \otimes_T t_2) \oplus_T (t_1 \otimes_T t_3)) \\
& = & (c \otimes_S b,\ t_1 \otimes_T (t_2 \oplus_T t_3))
\end{array}$$

We need to select t_1, t_2, t_3 so that

$$t_{\rm lhs} \equiv t_1 \otimes_T t_3 \neq t_1 \otimes_T (t_2 \oplus_T t_3) \equiv t_{\rm rhs}.$$

Case 3.2.1: $z \otimes_T x = z \otimes_T (x \oplus_T y)$. Then Then letting $t_1 = z$, $t_2 = x$, and $t_3 = y$ we have

$$t_{\rm lhs} = z \otimes_T y \neq z \otimes_T x = z \otimes_T (x \oplus_T y) = t_{\rm rhs}.$$

Case 3.2.2: $z \otimes_T x \neq z \otimes_T (x \oplus_T y)$. letting $t_1 = z$, $t_2 = y$, and $t_3 = x$ we have

$$t_{\rm lhs} = z \otimes_T x \neq z \otimes_T (x \oplus_T y) = z \otimes_T (y \oplus_T x) = t_{\rm rhs}.$$

$\square$

Computing Counter Examples

Note that from (a, b, c) such $c \otimes_S a = c \otimes_S b \wedge a \neq b$ and (x, y, z) such that $z \otimes_T x \neq z \otimes_T y$ our proof computes a counter example to LD as

$$\begin{array}{l}
\text{if } a = a \oplus_S b \\
\text{then if } z \otimes_T x = (z \otimes_T x) \oplus_T (z \otimes_T y) \\
\qquad \text{then } ((a,\ z),\ (b,\ y),\ (c,\ x)) \\
\qquad \text{else } ((a,\ z),\ (b,\ x),\ (c,\ y)) \\
\text{else if } z \otimes_T x = (z \otimes_T x) \oplus_T (z \otimes_T y) \\
\qquad \text{then } ((a,\ z),\ (b,\ x),\ (c,\ y)) \\
\qquad \text{else } ((a,\ z),\ (b,\ y),\ (c,\ x))
\end{array}$$

Examples

True or counter example

name	S	$\oplus$	$\otimes$	$\mathbb{LD}$	$\mathbb{LC}(S, \otimes)$	$\mathbb{LK}(S, \otimes)$
min_plus	$\mathbb{N}$	min	$+$	$\star$	$\star$	$(0, 0, 1)$
max_min	$\mathbb{N}$	max	min	$\star$	$(0, 0, 1)$	$(1, 0, 1)$

For example, $(0, 0, 1)$ is a counter example for $\mathbb{LC}(\mathbb{N}, \text{min})$ since $0 \,\text{min}\, 0 = 0 \,\text{min}\, 1$, but $0 \neq 1$.

Let's turn the crank

$$
\begin{array}{ll}
 & \mathbb{LD}(\text{min\_plus} \vec{\times} \text{max\_min}) \\
\Leftrightarrow & \mathbb{LD}(\text{min\_plus}) \wedge \mathbb{LD}(\text{max\_min}) \wedge (\mathbb{LC}(\mathbb{N},\ +) \vee \mathbb{LK}(\mathbb{N},\ \text{min})) \\
\Leftrightarrow & \text{TRUE}
\end{array}
$$

Examples

Another turn of the crank

$$
\begin{array}{ll}
 & \mathbb{LD}(\text{max\_min} \vec{\times} \text{min\_plus}) \\
\Leftrightarrow & \mathbb{LD}(\text{max\_min}) \wedge \mathbb{LD}(\text{min\_plus}) \wedge (\mathbb{LC}(\mathbb{N},\ \text{min}) \vee \mathbb{LK}(\mathbb{N},\ +)) \\
\Leftrightarrow & \text{FALSE}
\end{array}
$$

Note that the counter examples to $\mathbb{LC}$ and $\mathbb{LK}$ can be plugged into the proof above to produce the a counter example to $\mathbb{LD}$,

$$((0,\ 0),\ (0,\ 0),\ (1,\ 1))$$

and sure enough, with $\oplus = \text{max} \vec{\times} \text{min}$ and $\otimes = \text{min} \times +$ we have

$$(0,\ 0) \otimes ((0,\ 0) \oplus (1,\ 1)) = (0,\ 0) \otimes (1,\ 1) = (0,\ 1)$$

but

$$((0,\ 0) \otimes (0,\ 0)) \oplus ((0,\ 0) \otimes (1,\ 1)) = (0,\ 0) \oplus (0,\ 1) = (0,\ 0)$$

Another construction

Suppose that $(S, \oplus_S)$ and $(T, \oplus_T)$ are both commutative and idempotent semigroups. Recall that $S \uplus T$ represents the disjoint union of sets S and T. That is,

$$S \uplus T \equiv \{\mathrm{inl}(s) \mid s \in S\} \cup \{\mathrm{inr}(t) \mid t \in T\}.$$

Define the operation $\oplus \equiv \oplus_S + \oplus_T$ over $S \uplus T$ as

$$\begin{array}{rcl} \mathrm{inl}(s) \oplus \mathrm{inl}(s') & \equiv & \mathrm{inl}(s \oplus_S s') \\ \mathrm{inr}(t) \oplus \mathrm{inr}(t') & \equiv & \mathrm{inr}(t \oplus_T t') \\ \mathrm{inl}(s) \oplus \mathrm{inr}(t) & \equiv & \mathrm{inl}(s) \\ \mathrm{inr}(t) \oplus \mathrm{inl}(s) & \equiv & \mathrm{inl}(s) \end{array}$$

Homework 1: Due 1 November

Recall

$$S \uplus T \equiv \{\mathrm{inl}(s) \mid s \in S\} \cup \{\mathrm{inr}(t) \mid t \in T\}$$

Suppose that $(S, \oplus_S, \otimes_S)$ and $(T, \oplus_T, \otimes_T)$. are two semirings.

1. We want to define a combinator (combinators?) to combine these semirings to produce a semiring of the form

$$(S \uplus T, \oplus, \otimes).$$

   Explore ways in which you can define $\oplus$ and $\otimes$.
2. Can you give an informal interpretation for the resulting semiring(s)?
3. Present a network configuration using the above.