Lecture 8

Semantics of STLC terms in a ccc

Given a cartesian closed category $\mathbf{C}$,

given any function M mapping

- ground types G to $\mathbf{C}$-objects $M(G)$
  (which extends to a function mapping all types to objects, $A \mapsto M[\![A]\!]$, as we have seen)

Semantics of STLC terms in a ccc

Given a cartesian closed category $\mathbf{C}$,

given any function M mapping

- ground types G to $\mathbf{C}$-objects $M(G)$
- constants c^A to $\mathbf{C}$-morphisms $M(c^A) : 1 \to M[\![A]\!]$
  (In a category with a terminal object 1, given an object $X \in \mathbf{C}$, morphisms $1 \to X$ are sometimes called global elements of X.)

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we get a function mapping provable instances of the typing relation $\Gamma \vdash t : A$ to $\mathbf{C}$-morphisms

$$\boxed{M[\![\Gamma \vdash t : A]\!] : M[\![\Gamma]\!] \to M[\![A]\!]}$$

defined by recursing over the proof of $\Gamma \vdash t : A$ from the typing rules (which follows the structure of t):

Semantics of STLC terms in a ccc

Variables:

$$M[\![\Gamma, x : A \vdash x : A]\!] = M[\![\Gamma]\!] \times M[\![A]\!] \xrightarrow{\pi_2} M[\![A]\!]$$

$$M[\![\Gamma, x' : A' \vdash x : A]\!] =$$

$$M[\![\Gamma]\!] \times M[\![A']\!] \xrightarrow{\pi_1} M[\![\Gamma]\!] \xrightarrow{M[\![\Gamma \vdash x : A]\!]} M[\![A]\!]$$

Constants:

$$M[\![\Gamma \vdash c^A : A]\!] = M[\![\Gamma]\!] \xrightarrow{\langle\rangle} 1 \xrightarrow{M(c^A)} M[\![A]\!]$$

Unit value:

$$M[\![\Gamma \vdash () : \mathtt{unit}]\!] = M[\![\Gamma]\!] \xrightarrow{\langle\rangle} 1$$

Semantics of STLC terms in a ccc

Pairing:

$$M[\![\Gamma \vdash (s\,,\,t) : A \times B]\!] =$$

$$M[\![\Gamma]\!] \xrightarrow{\langle M[\![\Gamma \vdash s:A]\!], M[\![\Gamma \vdash t:B]\!]\rangle} M[\![A]\!] \times M[\![B]\!]$$

Projections:

$$M[\![\Gamma \vdash \mathtt{fst}\; t : A]\!] =$$

$$M[\![\Gamma]\!] \xrightarrow{M[\![\Gamma \vdash t:A \times B]\!]} M[\![A]\!] \times M[\![B]\!] \xrightarrow{\pi_1} M[\![A]\!]$$

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Projections:

Given that $\Gamma \vdash \texttt{fst}\, t : A$ holds, there is a unique type B such that $\Gamma \vdash t : A \times B$ already holds.

$$M[\![\Gamma \vdash \texttt{fst}\, t : A]\!] =$$

$$M[\![\Gamma]\!] \xrightarrow{M[\![\Gamma\vdash t:A\times B]\!]} M[\![A]\!] \times M[\![B]\!] \xrightarrow{\pi_1} M[\![A]\!]$$

Lemma. If $\Gamma \vdash t : A$ and $\Gamma \vdash t : B$ are provable, then $A = B$.

Semantics of STLC terms in a ccc

Pairing:

$$M[\![\Gamma \vdash (s\,,\,t) : A \times B]\!] =$$

$$M[\![\Gamma]\!] \xrightarrow{\langle M[\![\Gamma\vdash s:A]\!], M[\![\Gamma\vdash t:B]\!]\rangle} M[\![A]\!] \times M[\![B]\!]$$

Projections:

$$M[\![\Gamma \vdash \texttt{snd}\, t : B]\!] =$$

$$M[\![\Gamma]\!] \xrightarrow{M[\![\Gamma\vdash t:A\times B]\!]} M[\![A]\!] \times M[\![B]\!] \xrightarrow{\pi_2} M[\![B]\!]$$

(As for the case of `fst`, if $\Gamma \vdash \texttt{snd}\, t : B$, then $\Gamma \vdash t : A \times B$ already holds for a unique type A.)

Semantics of STLC terms in a ccc

Function abstraction:

$$M[\![\Gamma \vdash \lambda x : A.t : A \text{->} B]\!] = \mathtt{cur}\, f : M[\![\Gamma]\!] \longrightarrow (M[\![A]\!] \to M[\![B]\!])$$

where

$$f = M[\![\Gamma, x : A \vdash t : B]\!] : M[\![\Gamma]\!] \times M[\![A]\!] \longrightarrow M[\![B]\!]$$

Semantics of STLC terms in a ccc

Function application:

$$M[\![\Gamma \vdash s\,t : B]\!] =$$

$$M[\![\Gamma]\!] \xrightarrow{\langle f,g\rangle} (M[\![A]\!] \to M[\![B]\!]) \times M[\![A]\!] \xrightarrow{\texttt{app}} M[\![B]\!]$$

where

$A =$ unique type such that $\Gamma \vdash s : A \mathbin{\text{->}} B$ and $\Gamma \vdash t : A$ already holds (exists because $\Gamma \vdash s\,t : B$ holds)

$f = M[\![\Gamma \vdash s : A \mathbin{\text{->}} B]\!] : M[\![\Gamma]\!] \longrightarrow (M[\![A]\!] \to M[\![B]\!])$

$g = M[\![\Gamma \vdash t : A]\!] : M[\![\Gamma]\!] \longrightarrow M[\![A]\!]$

Example

Consider $\boxed{t \triangleq \lambda x : A.\, g(f\ x)}$ so that $\Gamma \vdash t : A \to C$ when $\Gamma \triangleq \diamond, f : A \to B, g : B \to C$.

Suppose $M[\![A]\!] = X$, $M[\![B]\!] = Y$ and $M[\![C]\!] = Z$ in $\mathbf{C}$. Then

$$M[\![\Gamma]\!] = (1 \times Y^X) \times Z^Y$$
$$M[\![\Gamma, x : A]\!] = ((1 \times Y^X) \times Z^Y) \times X$$

$$M[\![\Gamma, x : A \vdash x : A]\!] = \pi_2$$
$$M[\![\Gamma, x : A \vdash g : B \to C]\!] = \pi_2 \circ \pi_1$$
$$M[\![\Gamma, x : A \vdash f : A \to B]\!] = \pi_2 \circ \pi_1 \circ \pi_1$$
$$M[\![\Gamma, x : A \vdash f\ x : B]\!] = \mathrm{app} \circ \langle \pi_2 \circ \pi_1 \circ \pi_1 \,,\, \pi_2 \rangle$$
$$M[\![\Gamma, x : A \vdash g\ (f\ x) : C]\!] = \mathrm{app} \circ \langle \pi_2 \circ \pi_1 \,,\, \mathrm{app} \circ \langle \pi_2 \circ \pi_1 \circ \pi_1 \,,\, \pi_2 \rangle \rangle$$
$$M[\![\Gamma \vdash t : A \to C]\!] = \mathrm{cur}(\mathrm{app} \circ \langle \pi_2 \circ \pi_1 \,,\, \mathrm{app} \circ \langle \pi_2 \circ \pi_1 \circ \pi_1 \,,\, \pi_2 \rangle \rangle)$$

STLC equations

take the form $\boxed{\Gamma \vdash s = t : A}$ where $\Gamma \vdash s : A$ and $\Gamma \vdash t : A$ are provable.

Such an equation is satisfied by the semantics in a ccc if $M[\![\Gamma \vdash s : A]\!]$ and $M[\![\Gamma \vdash t : A]\!]$ are equal $\mathbf{C}$-morphisms $M[\![\Gamma]\!] \longrightarrow M[\![A]\!]$.

Qu: which equations are always satisfied in any ccc?

STLC equations

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Qu: which equations are always satisfied in any ccc?

Ans: $(\alpha)\beta\eta$-equivalence — to define this, first have to define alpha-equivalence, substitution and its semantics.

Alpha equivalence of STLC terms

The names of λ-bound variables should not affect meaning.

E.g. $\lambda f : A \rightarrow B.\, \lambda x : A.\, f\, x$ should have the same meaning as $\lambda x : A \rightarrow B.\, \lambda y : A.\, x\, y$

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This issue is best dealt with at the level of syntax rather than semantics: from now on we re-define "STLC term" to mean not an abstract syntax tree (generated as described before), but rather an equivalence class of such trees with respect to alpha-equivalence $\boxed{s =_\alpha t}$, defined as follows…

(Alternatively, one can use a "nameless" (de Bruijn) representation of terms.)

Alpha equivalence of STLC terms

$$\frac{}{c^A =_\alpha c^A}$$

$$\frac{}{x =_\alpha x}$$

$$\frac{}{() =_\alpha ()}$$

$$\frac{s =_\alpha s' \qquad t =_\alpha t'}{(s\,,t) =_\alpha (s'\,,t')}$$

$$\frac{t =_\alpha t'}{\texttt{fst}\,t =_\alpha \texttt{fst}\,t'}$$

$$\frac{t =_\alpha t'}{\texttt{snd}\,t =_\alpha \texttt{snd}\,t'}$$

$$\frac{s =_\alpha s' \qquad t =_\alpha t'}{s\,t =_\alpha s'\,t'}$$

$$\frac{(y\ x) \cdot t =_\alpha (y\ x') \cdot t' \qquad y \text{ does not occur in } \{x, x', t, t'\}}{\lambda x : A.\, t =_\alpha \lambda x' : A.\, t'}$$

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$$\frac{t =_\alpha t'}{\texttt{fst}\; t =_\alpha \texttt{fst}\; t'}$$

$$\frac{t =_\alpha t'}{\texttt{snd}\; t =_\alpha \texttt{snd}\; t'}$$

$$\frac{s =_\alpha s' \qquad t =_\alpha t'}{s\,t =_\alpha s'\,t'}$$

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result of replacing all occurrences of x with y in t

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$$\frac{s =_\alpha s' \qquad t =_\alpha t'}{(s\,,t) =_\alpha (s'\,,t')}$$

$$\frac{t =_\alpha t'}{\mathtt{fst}\ t =_\alpha \mathtt{fst}\ t'}$$

$$\frac{t =_\alpha t'}{\mathtt{snd}\ t =_\alpha \mathtt{snd}\ t'}$$

$$\frac{s =_\alpha s' \qquad t =_\alpha t'}{s\ t =_\alpha s'\,t'}$$

$$\frac{(y\ x) \cdot t =_\alpha (y\ x') \cdot t' \qquad y \text{ does not occur in } \{x, x', t, t'\}}{\lambda x : A.\, t =_\alpha \lambda x' : A.\, t'}$$

E.g.

$\lambda x : A.\, x\, x =_\alpha \lambda y : A.\, y\, y \neq_\alpha \lambda x : A.\, x\, y$

$(\lambda y : A.\, y)\, x =_\alpha (\lambda x : A.\, x)\, x \neq_\alpha (\lambda x : A.\, x)\, y$

Substitution

$t[s/x]$ = result of replacing all free occurrences of variable x in term t (i.e. those not occurring within the scope of a $\lambda x : A.\_$ binder) by the term s, alpha-converting λ-bound variables in t to avoid them "capturing" any free variables of t.

E.g. $(\lambda y : A.\,(y\,,\,x))[y/x]$ is $\lambda z : A.\,(z\,,\,y)$ and is not $\lambda y : A.\,(y\,,\,y)$

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The relation $t[s/x] = t'$ can be inductively defined by the following rules...

Substitution

$$\frac{}{c^A[s/x] = c^A}$$

$$\frac{}{x[s/x] = s}$$

$$\frac{y \neq x}{y[s/x] = y}$$

$$\frac{}{()[s/x] = ()}$$

$$\frac{t_1[s/x] = t_1' \qquad t_2[s/x] = t_2'}{(t_1\,,t_2)[s/x] = (t_1'\,,t_2')}$$

$$\frac{t[s/x] = t'}{(\mathtt{fst}\;t)[s/x] = \mathtt{fst}\;t'}$$

$$\frac{t[s/x] = t'}{(\mathtt{snd}\;t)[s/x] = \mathtt{snd}\;t'}$$

$$\frac{t_1[s/x] = t_1' \qquad t_2[s/x] = t_2'}{(t_1\;t_2)[s/x] = t_1'\,t_2'}$$

$$\frac{t[s/x] = t' \qquad y \neq x \text{ and } y \text{ does not occur in } s}{(\lambda y : A.\, t)[s/x] = \lambda y : A.\, t'}$$

Semantics of substitution in a ccc

Substitution Lemma If $\Gamma \vdash s : A$ and $\Gamma, x : A \vdash t : B$ are provable, then so is $\Gamma \vdash t[s/x] : B$.

Substitution Theorem If $\Gamma \vdash s : A$ and $\Gamma, x : A \vdash t : B$ are provable, then in any ccc the following diagram commutes:

$$\begin{array}{ccc} M[\![\Gamma]\!] & \xrightarrow{\langle \mathtt{id}, M[\![\Gamma \vdash s : A]\!]\rangle} & M[\![\Gamma]\!] \times M[\![A]\!] \\ & \searrow^{M[\![\Gamma \vdash t[s/x] : B]\!]} & \downarrow{\scriptstyle M[\![\Gamma, x : A \vdash t : B]\!]} \\ & & M[\![B]\!] \end{array}$$