Complexity Theory

Lecture 4

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http://www.cl.cam.ac.uk/teaching/1920/Complexity

Verifiers

A verifier V for a language L is an algorithm such that

$$L = \{x \mid (x, c) \text{ is accepted by } V \text{ for some } c\}$$

If V runs in time polynomial in the length of x, then we say that

L is polynomially verifiable.

Many natural examples arise, whenever we have to construct a solution to some design constraints or specifications.

Nondeterminism

If, in the definition of a Turing machine, we relax the condition on δ being a function and instead allow an arbitrary relation, we obtain a *nondeterministic Turing machine*.

$$\delta \subseteq (Q \times \Sigma) \times (Q \cup \{\mathrm{acc}, \mathrm{rej}\} \times \Sigma \times \{R, L, S\}).$$

The yields relation $\rightarrow_M$ is also no longer functional.

We still define the language accepted by M by:

$$\{x \mid (s, \triangleright, x) \rightarrow_M^\star (\mathrm{acc}, w, u) \text{ for some } w \text{ and } u\}$$

though, for some x, there may be computations leading to accepting as well as rejecting states.

Computation Trees

With a nondeterministic machine, each configuration gives rise to a tree of successive configurations.

$$\begin{tikzcd}
& (s, \triangleright, x) \arrow[dl] \arrow[d] \arrow[dr] & \\
(q_0, u_0, w_0) \arrow[d] & (q_1, u_1, w_1) \arrow[dd] \arrow[ddr] & (q_2, u_2, w_2) \arrow[d] \\
(q_{00}, u_{00}, w_{00}) & & (\mathrm{rej}, u_2, w_2) \\
& (q_{10}, u_{10}, w_{10}) & (q_{11}, u_{11}, w_{11}) \\
\vdots & & \vdots \\
(\mathrm{acc}, \ldots) & &
\end{tikzcd}$$

Nondeterministic Complexity Classes

We have already defined TIME(f) and SPACE(f).

NTIME(f) is defined as the class of those languages L which are accepted by a *nondeterministic* Turing machine M, such that for every $x \in L$, there is an accepting computation of M on x of length $O(f(n))$, where n is the length of x.

$$\mathrm{NP} = \bigcup_{k=1}^{\infty} \mathrm{NTIME}(n^k)$$

Nondeterminism

$$\begin{tikzcd}
 & (s, \triangleright, x) \arrow[dl] \arrow[d] \arrow[dr] & & \\
(q_0, u_0, w_0) \arrow[d] & (q_1, u_1, w_1) \arrow[dd] \arrow[ddrr] & (q_2, u_2, w_2) \arrow[dr] & \\
(q_{00}, u_{00}, w_{00}) & & & (\mathrm{rej}, u_2, w_2) \\
 & (q_{10}, u_{10}, w_{10}) & & (q_{11}, u_{11}, w_{11}) \\
\vdots & & & \vdots \\
(\mathrm{acc}, \ldots) & & &
\end{tikzcd}$$

For a language in NTIME(f), the height of the tree can be bounded by $f(n)$ when the input is of length n.

NP

A language L is polynomially verifiable if, and only if, it is in NP.

To prove this, suppose L is a language, which has a verifier V, which runs in time $p(n)$.

The following describes a *nondeterministic algorithm* that accepts L

1. input x of length n
2. nondeterministically guess c of length $\leq p(n)$
3. run V on (x, c)

NP

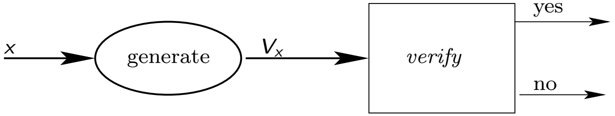
In the other direction, suppose M is a nondeterministic machine that accepts a language L in time n^k.

We define the *deterministic algorithm* V which on input (x, c) simulates M on input x.
At the i^{th} nondeterministic choice point, V looks at the i^{th} character in c to decide which branch to follow.
If M accepts then V accepts, otherwise it rejects.

V is a polynomial verifier for L.

Generate and Test

We can think of nondeterministic algorithms in the generate-and test paradigm:



Where the *generate* component is nondeterministic and the *verify* component is deterministic.

Reductions

Given two languages $L_1 \subseteq \Sigma_1^\star$, and $L_2 \subseteq \Sigma_2^\star$,

A *reduction* of L_1 to L_2 is a *computable* function

$$f : \Sigma_1^\star \to \Sigma_2^\star$$

such that for every string $x \in \Sigma_1^\star$,

$$f(x) \in L_2 \text{ if, and only if, } x \in L_1$$

Resource Bounded Reductions

If f is computable by a polynomial time algorithm, we say that L_1 is *polynomial time reducible* to L_2.

$$L_1 \leq_P L_2$$

If f is also computable in $\mathsf{SPACE}(\log n)$, we write

$$L_1 \leq_L L_2$$

Reductions 2

If $L_1 \leq_P L_2$ we understand that L_1 is no more difficult to solve than L_2, at least as far as polynomial time computation is concerned.

That is to say,

If $L_1 \leq_P L_2$ and $L_2 \in \mathsf{P}$, then $L_1 \in \mathsf{P}$

We can get an algorithm to decide L_1 by first computing f, and then using the polynomial time algorithm for L_2.

Completeness

The usefulness of reductions is that they allow us to establish the *relative* complexity of problems, even when we cannot prove absolute lower bounds.

Cook (1972) first showed that there are problems in NP that are maximally difficult.

A language L is said to be NP-*hard* if for every language $A \in \mathsf{NP}$, $A \leq_P L$.

A language L is NP-*complete* if it is in NP and it is NP-hard.