Complexity Theory

Lecture 7

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http://www.cl.cam.ac.uk/teaching/1920/Complexity

Sets, Numbers and Scheduling

It is not just problems about formulas and graphs that turn out to be NP-complete.

Literally hundreds of naturally arising problems have been proved NP-complete, in areas involving network design, scheduling, optimisation, data storage and retrieval, artificial intelligence and many others.

Such problems arise naturally whenever we have to construct a solution within constraints, and the most effective way appears to be an exhaustive search of an exponential solution space.

We now examine three more NP-complete problems, whose significance lies in that they have been used to prove a large number of other problems NP-complete, through reductions.

3D Matching

The decision problem of *3D Matching* is defined as:

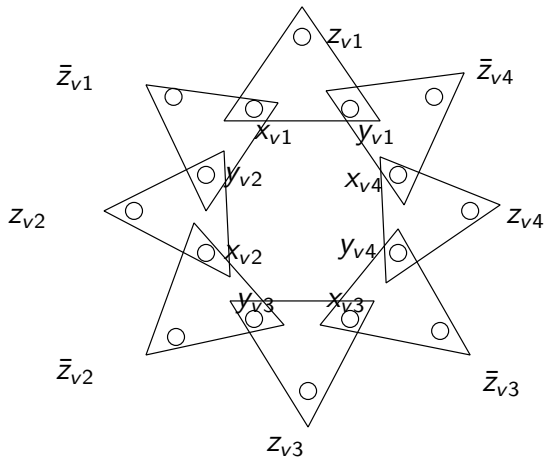> *Given three disjoint sets X, Y and Z, and a set of triples*
> *$M \subseteq X \times Y \times Z$, does M contain a matching?*
> *I.e. is there a subset $M' \subseteq M$, such that each element of X, Y*
> *and Z appears in exactly one triple of M'?*

We can show that 3DM is NP-complete by a reduction from 3SAT.

Reduction

If a Boolean expression ϕ in 3CNF has n variables, and m clauses, we construct for each variable v the following gadget.

In addition, for every clause c, we have two elements x_c and y_c. If the literal v occurs in c, we include the triple

$$(x_c, y_c, z_{vc})$$

in M.

Similarly, if $\neg v$ occurs in c, we include the triple

$$(x_c, y_c, \bar{z}_{vc})$$

in M.

Finally, we include extra dummy elements in X and Y to make the numbers match up.

Exact Set Covering

Two other well known problems are proved NP-complete by immediate reduction from 3DM.

Exact Cover by 3-Sets is defined by:

> *Given a set U with $3n$ elements, and a collection*
> *$S = \{S_1, \ldots, S_m\}$ of three-element subsets of U, is there a*
> *sub-collection containing exactly n of these sets whose union is*
> *all of U?*

The reduction from 3DM simply takes $U = X \cup Y \cup Z$, and S to be the collection of three-element subsets resulting from M.

Set Covering

More generally, we have the *Set Covering* problem:

> *Given a set U, a collection $S = \{S_1, \ldots, S_m\}$ of subsets of U and an integer budget B, is there a collection of B sets in S whose union is U?*

Knapsack

KNAPSACK is a problem which generalises many natural scheduling and optimisation problems, and through reductions has been used to show many such problems NP-complete.

In the problem, we are given n items, each with a positive integer value v_i and weight w_i.
We are also given a maximum total weight W, and a minimum total value V.

> *Can we select a subset of the items whose total weight does not exceed W, and whose total value is at least V?*

Reduction

The proof that KNAPSACK is NP-complete is by a reduction from the problem of Exact Cover by 3-Sets.

Given a set $U = \{1, \ldots, 3n\}$ and a collection of 3-element subsets of U, $S = \{S_1, \ldots, S_m\}$.

We map this to an instance of KNAPSACK with m elements each corresponding to one of the S_i, and having weight and value

$$\sum_{j \in S_i} (m+1)^{j-1}$$

and set the target weight and value both to

$$\sum_{j=0}^{3n-1} (m+1)^j$$

Scheduling

Some examples of the kinds of scheduling tasks that have been proved NP-complete include:

Timetable Design

> Given a set H of *work periods*, a set W of *workers* each with an associated subset of H (available periods), a set T of *tasks* and an assignment $r : W \times T \to \mathbb{N}$ of *required work*, is there a mapping $f : W \times T \times H \to \{0, 1\}$ which completes all tasks?

Scheduling

Sequencing with Deadlines

Given a set T of tasks and for each task a length $l \in \mathbb{N}$, a release time $r \in \mathbb{N}$ and a deadline $d \in \mathbb{N}$, is there a work schedule which completes each task between its release time and its deadline?

Job Scheduling

Given a set T of tasks, a number $m \in \mathbb{N}$ of processors a length $l \in \mathbb{N}$ for each task, and an overall deadline $D \in \mathbb{N}$, is there a multi-processor schedule which completes all tasks by the deadline?