4 Data Science (DJW)

I am playing a game of solitaire, which involves repeatedly tossing a fair coin. If I get three heads in a row I win, if I get two tails in a row I lose.

(*a*) Devise a Markov chain to represent the state of the game, and draw the state space. The state space diagram should have eight states, including

- a state $\varnothing$ to represent "not yet tossed any coins",
- a state TT to represent "lost", with a single outgoing transition back to state TT,
- a state HHH to represent "won", with a single outgoing transition back to state HHH.

[6 marks]

(*b*) I wish to compute the probability of winning. Let ρ_x be the probability that I will win, when starting from state x. Clearly $\rho_{TT} = 0$ and $\rho_{HHH} = 1$. Show that for any other state x

$$\rho_x = \sum_y \mathbb{P}\big(\text{will win} \mid \text{start at } y\big) P_{xy}$$

for a suitable matrix P, which you should define. Explain your reasoning carefully. [5 marks]

(*c*) Write out a set of equations that could be solved to find $\rho_\varnothing$. You do not need to solve them. [1 mark]

(*d*) Explain what is meant by *stationary distribution*. [2 marks]

(*e*) Let $\lambda \in [0, 1]$, and define a distribution π by

$$\pi_x = \lambda 1_{x=TT} + (1-\lambda) 1_{x=HHH}\,.$$

Show that π is a stationary distribution for your Markov chain. [6 marks]