Quantum Computing: Exercise Sheet 2

Steven Herbert and Anuj Dawar

1. Let $|\psi\rangle = \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle$. Show by explicitly expressing the tensor product that:
$$(X \otimes I)|\psi\rangle = \alpha|10\rangle + \beta|11\rangle + \gamma|00\rangle + \delta|01\rangle$$

2. Let A and B be 2×2 matrices, and $|\psi\rangle$ and $|\phi\rangle$ be single qubit states (i.e., 2×1 element vectors). Show that:
$$(A \otimes B)(|\psi\rangle \otimes |\phi\rangle) = (A|\psi\rangle) \otimes (B|\phi\rangle)$$

3. Show that a swap gate can be constructed from three CNOT gates.

4. Give the matrix form of the controlled-Hadamard gate. That is a two-qubit gate – let the first qubit be the control and the second be the target – if the first (control) qubit equals zero, then the second (target) qubit is unchanged; if the first (control) qubit equals one, then a Hadamard gate is performed on the second (target) qubit.

   Hint: Look at the matrix form of the CNOT gate, in which the first qubit controls a Pauli-X (not) gate on the second qubit.

5. Show how the controlled-Not gate can be constructed from Hadamard gates and the controlled-Z. Demonstrate that the construction is correct by multiplying the corresponding matrices.

6. Verify that the four Bell states ($\frac{1}{\sqrt{2}}(|00\rangle+|11\rangle)$, $\frac{1}{\sqrt{2}}(|00\rangle-|11\rangle)$, $\frac{1}{\sqrt{2}}(|01\rangle+|10\rangle)$, $\frac{1}{\sqrt{2}}(|01\rangle-|10\rangle)$) form an orthonormal basis for $\mathbb{C}^4$.

7. Suppose that Eve intercepts the qubit transmitted by Alice in the superdense coding protocol. Can she infer which of the four pairs of bits 00; 01; 10, or 11 Alice was trying to transmit? If so, how? If not, why not?

8. Let Alice and Bob each have one qubit of a Bell pair ($\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$); let Bob and Charlie also each have one half of another Bell pair. If Bob uses the Bell pair he shares with Charlie, to teleport his qubit from the Bell pair he (Bob) shares with Alice, show that the result is that Alice and Charlie now share a Bell pair.

9. Alice and Bob are using the BB84 protocol, and Eve attempts the intercept, measure and retransmit attach described in the lectures. If Alice and Bob compare a bit-string of length n (i.e., from their shared channel) to determine whether they are being eavesdropped, what is the probability that Eve's actions remain undetected?

10. Regarding the statements on Slides 14 and 15 of lecture 7, show that:

    (a) The Unitary in the Deutsch-Jozsa algorithm indeed transforms:
$$|\psi_1\rangle = \sum_{x\in\{0,1\}^n} \frac{1}{\sqrt{2^n}}|x\rangle\left(|0\rangle - |1\rangle\right).$$

to:

$$|\psi_2\rangle = \sum_{x\in\{0,1\}^n} \frac{1}{\sqrt{2^n}}(-1)^{f(x)}|x\rangle\left(|0\rangle - |1\rangle\right).$$

(b) $H^{\otimes n}|0\rangle^{\otimes n} = \frac{1}{\sqrt{2^n}}\sum_{x\in\{0,1\}^n}|x\rangle$,

(c) $H^{\otimes n}|x\rangle = \frac{1}{\sqrt{2^n}}\sum_{z\in\{0,1\}^n}(-1)^{x\cdot z}|z\rangle$.

11. Let $n = 2$ and

    (a) $f(x) = 0$ if x is even or zero, and 1 otherwise,

    (b) $f(x) = 0$ for all x.

    For each case, work through the Deutsch-Jozsa algorithm explicitly, and show the probabilities for each of the $2^n = 4$ possible measurements of the two qubits.

12. In Lecture 8 it was claimed that $C_{n-1}X$ (that is, a Pauli-X gate controlled by the "And" of $n-1$ qubits) could be implemented efficiently using Toffoli gates and some extra workspace qubits. Show how $C_{n-1}X$ can be constructed from $n-2$ Toffoli gates and $n-3$ workspace qubits (all in the state $|0\rangle$).

13. Suppose we apply Grover's algorithm to a 3 qubit register, in which only the state $|010\rangle$ is marked. What is the probability of measuring the marked state ($|010\rangle$) after applying the Grover iterate 0,1,2,3 times?