Topics in Logic and Complexity

Handout 5

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http://www.cl.cam.ac.uk/teaching/2122/L15

Constraint Satisfaction Problems

Example:

Can we find x, y, z such that

$$\begin{aligned} x + y + z &\geq 4 \\ x - y &= 3 \\ z &\leq 2 \\ x &= 1 \end{aligned}$$

Constraint Satisfaction Problems

In general a *constraint satisfaction problem (CSP)* is specified by:

- A collection V of *variables*.
- For each variable $x \in V$ a *domain* D_v of possible *values*.
- A collection of *constraints* each of which consists of a tuple $(x_1, \ldots, x_r)$ of variables and a set

$$S \subseteq D_{x_1} \times \cdots \times D_{x_r}$$

  of permitted combinations of values.

We consider *finite-domain* CSP, where the sets D_x are *finite*.
We further make the simplifying assumption that there is a *single domain* D, with $D_x = D$ for all $x \in V$.

Constraint Satisfaction Problems

In general a *constraint satisfaction problem (CSP)* is specified by:

- A collection V of *variables*.
- A domain D of *values*
- A collection of *constraints* each of which consists of a tuple $(x_1, \ldots, x_r)$ of variables and a set $S \subseteq D^r$ of permitted combinations of values.

The problem is to *decide* if there is an assignment

$$\eta : V \to D$$

such that for each constraint $C = (\mathrm{x}, S)$ we have

$$\eta(\mathrm{x}) \in S.$$

Example - Boolean Satisfiability

Consider a Boolean formula ϕ in *conjunctive normal form* (CNF).
This can be seen as *CSP* with

- V the set of variables occurring in ϕ
- $D = \{0, 1\}$
- a *constraint* for each *clause* of ϕ.

The clause $x \vee y \vee \bar{z}$ gives the constraint $(x, y, z), S$ where

$$S = \{(0,0,0),(0,1,0),(0,1,1),(1,0,0),(1,0,1),(1,1,0),(1,1,1)\}$$

Structure Homomorphism

Fix a relational signature σ (no function or constant symbols).
Let $\mathbb{A}$ and $\mathbb{B}$ be two σ-structures.
A *homomorphism* from $\mathbb{A}$ to $\mathbb{B}$ is a function $h : A \to B$ such that for each relation $R \in \sigma$ and each tuple $\mathbf{a}$

$$\mathbf{a} \in R^{\mathbb{A}} \quad \Rightarrow \quad h(\mathbf{a}) \in R^{\mathbb{B}}$$

The problem of deciding, given $\mathbb{A}$ and $\mathbb{B}$ whether there is a homomorphism from $\mathbb{A}$ to $\mathbb{B}$ is NP-complete. Why?

Homomorphism and CSP

Given a CSP with variables V, domain D and constraints $\mathcal{C}$, let σ be a signature with a relation symbol R_S of arity r for each distinct relation $S \subseteq D^r$ occurring in $\mathcal{C}$.

Let $\mathbb{B}$ be the σ-structure with universe D where each R_S is interpreted by the relation S

Let $\mathbb{A}$ be the structure with universe V where R_S is interpreted as the set of all tuples $\mathbf{x}$ for which $(\mathbf{x}, S) \in \mathcal{C}$.

Then, the CSP is solvable *if, and only if,* there is a homomorphism from $\mathbb{A}$ to $\mathbb{B}$.

Complexity of CSP

Write $\mathbb{A} \longrightarrow \mathbb{B}$ to denote that *there is* a homomorphism from $\mathbb{A}$ to $\mathbb{B}$.

The problem of determining, given $\mathbb{A}$ and $\mathbb{B}$, whether $\mathbb{A} \longrightarrow \mathbb{B}$ is *NP-complete*, and can be decided in time $O(|B|^{|A|})$.

So, for a fixed structure $\mathbb{A}$, the problem of deciding membership in the set

$$\{\mathbb{B} \mid \mathbb{A} \longrightarrow \mathbb{B}\}$$

is in P.

Non-uniform CSP

On the other hand, for a fixed structure $\mathbb{B}$, we define the *non-uniform CSP* with template $\mathbb{B}$, written CSP($\mathbb{B}$) as the class of structures

$$\{\mathbb{A} \mid \mathbb{A} \longrightarrow \mathbb{B}\}$$

The complexity of CSP($\mathbb{B}$) depends on the particular structure $\mathbb{B}$.

The problem is always in NP. For some $\mathbb{B}$, it is in P and for others it is NP-complete

Example - 3-SAT

Let $\mathbb{B}$ be a structure with universe $\{0, 1\}$ and *eight* relations

$$R_{000}, R_{001}, R_{010}, R_{011}, R_{100}, R_{101}, R_{110}, R_{111}$$

where R_{ijk} is defined to be the relation

$$\{0, 1\}^3 \setminus \{(i, j, k)\}.$$

Then, $\mathrm{CSP}(\mathbb{B})$ is *essentially* the problem of determining satisfiability of Boolean formulas in *3-CNF*.

Example - 3-Colourability

Let K_n be the *complete* simple undirected graph on n vertices.

Then, an undirected simple graph is in $\mathrm{CSP}(K_3)$ *if, and only if,* it is *3-colourable*.

$\mathrm{CSP}(K_3)$ is NP-complete.

On the other hand, $\mathrm{CSP}(K_2)$ is in P.

Example - 3XOR-SAT

Let $\mathbb{B}$ be a structure with universe $\{0, 1\}$ and *two* ternary relations

$$R_0 \text{ and } R_1.$$

where R_i is the collection of triples $(x, y, z) \in \{0, 1\}^3$ such that

$$x + y + z \equiv i \pmod 2$$

Then, $\mathrm{CSP}(\mathbb{B})$ is *essentially* the problem of determining satisfiability of Boolean formulas in *3-XOR-CNF*.
This problem is in P.

Schaefer's theorem

Schaefer (1978) proved that if $\mathbb{B}$ is a structure on domain $\{0, 1\}$, then CSP($\mathbb{B}$) is in P if one of the following cases holds:

1. Each relation of $\mathbb{B}$ is 0-*valid*.
2. Each relation of $\mathbb{B}$ is 1-*valid*.
3. Each relation of $\mathbb{B}$ is *bijunctive*.
4. Each relation of $\mathbb{B}$ is *Horn*.
5. Each relation of $\mathbb{B}$ is *dual Horn*.
6. Each relation of $\mathbb{B}$ is *affine*.

In all other cases, CSP($\mathbb{B}$) is *NP-complete*.

Hell-Nešetřil theorem

Let *H* be a *simple, undirected graph*.

Hell and Nešetřil (1990) proved that CSP(H) is in P if one of the following holds

1. H is *edgeless*
2. H is *bipartite*

In all other cases, CSP(H) is *NP-complete*.

Feder-Vardi conjecture

Feder and Vardi (1993) conjectured that for *every* finite relational structure $\mathbb{B}$:

either $\mathrm{CSP}(\mathbb{B})$ *is in* P *or it is* NP*-complete.*

Ladner (1975) showed that for any *languages* L and K, if $L \leq_P K$ and $K \not\leq_P L$, then there is a language M with

$$L \leq_P M \leq_P K \text{ and } K \not\leq_P M \text{ and } M \not\leq_P L$$

Corollary: if $\mathrm{P} \neq \mathrm{NP}$ then there are problems in NP that are neither in P nor NP-complete.

Bulatov-Zhuk theorem

Bulatov and Zhuk (2017) independently proved the Feder-Vardi *dichotomy conjecture*.

The result came after a twenty-year development of the theory of CSP based on *universal algebra*.

The complexity of CSP($\mathbb{B}$) can be completely classified based on the identitites satisfied by the *algebra of polymorphisms* of the structure $\mathbb{B}$.