Topics in Logic and Complexity

Lecture 14

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Polymorphisms

For a pair of structures $\mathbb{A}$ and $\mathbb{B}$ over the same relational structure σ, we write $\mathbb{A} \times \mathbb{B}$ for their *Cartesian product*.
This is defined to be the σ-structure with universe $A \times B$ so that for any r-ary $R \in \sigma$:

$$((a_1, b_1), \ldots, (a_r, b_r)) \in R^{\mathbb{A} \times \mathbb{B}} \quad \text{if, and only if,}$$

$$(a_1, \ldots, a_r) \in R^{\mathbb{A}} \textit{ and } (b_1, \ldots, b_r) \in R^{\mathbb{B}}.$$

Note: we always have $\mathbb{A} \times \mathbb{B} \longrightarrow \mathbb{A}$ and $\mathbb{A} \times \mathbb{B} \longrightarrow \mathbb{B}$

Polymorphisms

We define the *kth power* of $\mathbb{B}$, written $\mathbb{B}^k$ to be the Cartesian product of $\mathbb{B}$ to itself.

For a structure $\mathbb{B}$, a *k-ary polymorphism* of $\mathbb{B}$ is a homomorphism

$$h : \mathbb{B}^k \longrightarrow \mathbb{B}$$

The collection of all polymorphisms of $\mathbb{B}$ forms an *algebraic structure* called the *clone of polymorphisms* of $\mathbb{B}$.

Algebraic properties of this clone *determine* the *complexity* of CSP($\mathbb{B}$).

CSP and MSO

For any fixed finite structure $\mathbb{B}$, the class of structures $\mathrm{CSP}(\mathbb{B})$ is definable in *existential MSO*.

Let $b_1, \ldots, b_n$ *enumerate* the elements of $\mathbb{B}$.

$$\begin{aligned} \exists X_1 \cdots \exists X_n \quad & \forall x \bigvee_i X_i(x) \land \\ & \forall x \bigwedge_{i \neq j} X_i(x) \to \neg X_j(x) \land \\ & \bigwedge_{R \in \sigma} \forall x_1 \cdots \forall x_r \big(R(x_1 \cdots x_r) \to \bigvee_{(b_{i_1} \cdots b_{i_r}) \in R^{\mathbb{B}}} \bigwedge_j X_{i_j}(x_j) \big) \end{aligned}$$

A structure $\mathbb{A}$ satisfies this sentence *if, and only if,* $\mathbb{A} \longrightarrow \mathbb{B}$.

k-local Consistency Algorithm

For a positive integer k we define an algorithm called the *k-consistency algorithm* for testing whether $\mathbb{A} \longrightarrow \mathbb{B}$.

Let S_0 be the collection of all *partial homomorphisms* $h : \mathbb{A} \hookrightarrow \mathbb{B}$ with *domain size* k.

Given a set $S \subseteq S_0$, say that $h \in S$ is *extendable* in S if

> *for each restriction g of h to $k-1$ elements and eacch $a \in A$, there is an $h' \in S$ that extends g and whose domain includes a.*

k-local Consistency Algorithm

The k-consistency algorithm can now be described as follows

1. $S := S_0$;
2. $S' := \{h \in S \mid h \text{ is extendable in } S\}$
3. if $S' = \emptyset$ then reject
4. else if $S' = S$ then accept
5. else goto 2.

If this algorithm rejects then $\mathbb{A} \not\longrightarrow \mathbb{B}$.
If the algorithm accepts, we can't be sure.

Bounded Width CSP

We say that $CSP(\mathbb{B})$ has *width k* if the k-consistency algorithm *correctly* determines for each $\mathbb{A}$ whether or not $\mathbb{A} \longrightarrow \mathbb{B}$.

We say that $CSP(\mathbb{B})$ has *bounded width* if there is some k such that it has width k.

Note: If $CSP(\mathbb{B})$ has bounded width, it is solvable in *polynomial time*.

$\mathrm{CSP}(K_2)$ has width 3.
$\mathrm{CSP}(K_3)$ has *unbounded* width.

Definability in LFP

If CSP($\mathbb{B}$) is of bounded width, there is a sentence of LFP that *defines* it.

The k-consistency algorithm is computing the *largest* set $S \subseteq S_0$ such that every $h \in S$ is extendable in S.
This can be defined as the *greatest fixed point* of an operator definable in *first-order logic*.

Exercise: prove it!

Fact: If CSP($\mathbb{B}$) is definable in LFP then it has *bounded width*.

Fact: There are $\mathbb{B}$ for which CSP($\mathbb{B}$) is in P, but not of bounded width.

Near-Unanimity Polymorphisms

For $k \geq 3$, a function $f : B^k \to B$ is said to be a *near-unanimity* (NU) function if for all $a, b \in B$

$$f(a, \dots, a, b) = f(a, \dots, b, a) = \dots = f(b, \dots, a, a) = a.$$

Say $\mathbb{B}$ has a *near-unanimity polymorphism* of arity k if there is a k-ary near-unanimity function that is a *polymorphism* of $\mathbb{B}$.

Fact: if $\mathbb{B}$ has a NU polymorphism of arity k then for every $l > k$, it has a NU polymorphism of arity l.

If $g : \mathbb{B}^k \to \mathbb{B}$ is a NU polymorphism, define

$$h(x_1, \dots, x_l) = g(x_1, \dots, x_k)$$

Near-Unanimity and Bounded Width

Theorem
If $\mathbb{B}$ has a NU polymorphism of arity k, then $\mathrm{CSP}(\mathbb{B})$ has *width* k.

Suppose S is a *non-empty* set of partial homomorphisms $h : \mathbb{A} \hookrightarrow \mathbb{B}$, each of which is *extendable* in S.

We can use this *and* the NU polymorphisms of $\mathbb{B}$ to construct a *total* homomorphism $g : \mathbb{A} \to \mathbb{B}$.

Weak Near-Unanimity

For $k \geq 3$, a function $f : B^k \to B$ is said to be a *weak near-unanimity* (WNU) function if for all$a, b \in B$

$$f(a, \dots, a, b) = f(a, \dots, b, a) = \dots = f(b, \dots, a, a).$$

Theorem
If $\mathbb{B}$ does not have *any* weak near-unanimity polymorphisms, then CSP($\mathbb{B}$) is NP-complete.

Theorem (Bulatov; Zhuk)
If $\mathbb{B}$ has a weak near-unanimity polymorphism of any arity, then CSP($\mathbb{B}$) is in P.