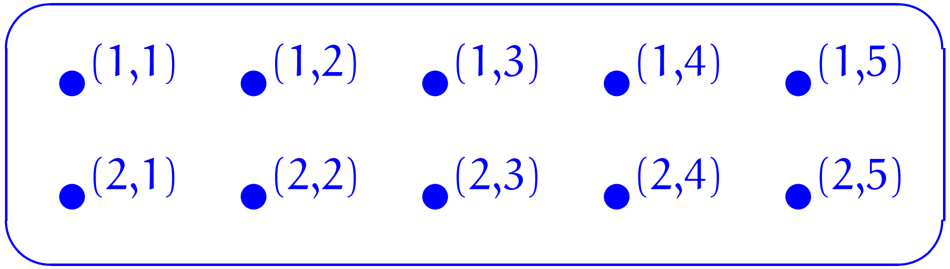
Sets

Objectives

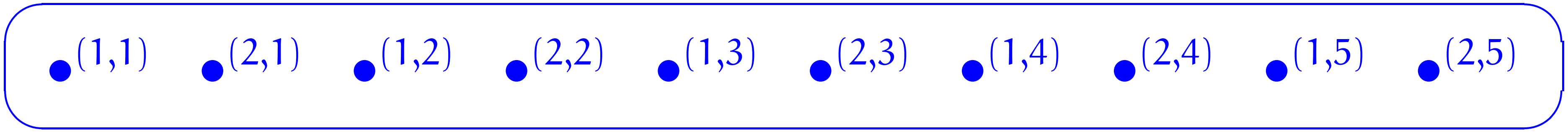
To introduce the basics of the theory of sets and some of its uses.

Abstract sets

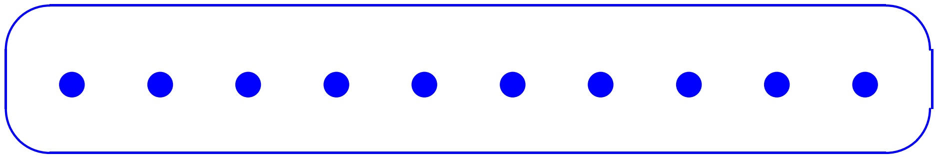
It has been said that a set is like a mental "bag of dots", except of course that the bag has no shape; thus,



may be a convenient way of picturing a certain set for some considerations, but what is apparently the same set may be pictured as



or even simply as



for other considerations.

Naive Set Theory

We are not going to be formally studying Set Theory here; rather, we will be *naively* looking at ubiquitous structures that are available within it.

Set membership

We write $\in$ for the *membership predicate*; so that

$$x \in A \text{ stands for } x \text{ is an element of } A \ .$$

We further write

$$x \notin A \text{ for } \neg(x \in A) \ .$$

Example: $0 \in \{0, 1\}$ and $1 \notin \{0\}$ are true statements.

Extensionality axiom

$\boxed{\text{Two sets are equal if they have the same elements.}}$

Thus,

$$\forall \text{ sets } A, B.\ A = B \iff (\forall x.\, x \in A \iff x \in B) \quad .$$

Example:

$$\{0\} \neq \{0,1\} = \{1,0\} \neq \{2\} = \{2,2\}$$

NB: $\{x,y\}$ is defined $\forall z \in \{x,y\} \Leftrightarrow (z=x \vee z=y)$

Proposition 100 *For* $b, c \in \mathbb{R}$, *let*

$$
\begin{aligned}
A &= \{ x \in \mathbb{C} \mid x^2 - 2bc + c = 0 \} \\
B &= \{ b + \sqrt{b^2 - c}, b - \sqrt{b^2 - c} \} \\
C &= \{ b \}
\end{aligned}
$$

ERRATA

$$B = C \Leftrightarrow \left(\begin{array}{l} b + \sqrt{b^2 - c} = b \\ \cancel{\vee} \wedge \; b - \sqrt{b^2 - c} = b \end{array} \right)$$

$$\Leftrightarrow b^2 = c.$$

Then,

1. $A = B$, *and*

2. $B = C \iff b^2 = c$.

- $B = C \Rightarrow b^2 = c$

Assume $B = C$. That is,

$$\{b+\sqrt{b^2-c},\ b-\sqrt{b^2-c}\} = \{b\} \Rightarrow b$$

$$b \in \{b+\sqrt{b^2-c},\ b-\sqrt{b^2-c}\}$$

$$\Rightarrow \left.\begin{array}{l} b = b+\sqrt{b^2-c} \\ \vee \\ b = b-\sqrt{b^2-c} \end{array}\right] \Rightarrow b^2 = c$$

$$\left\{b-\sqrt{b^2-c},\ b+\sqrt{b^2-c}\right\}=\{b\}$$

$\Rightarrow$

$$b-\sqrt{b^2-c}\in\{b\}\Rightarrow b-\sqrt{b^2-c}=b$$

also

$$b+\sqrt{b^2-c}\in\{b\}\Rightarrow b+\sqrt{b^2-c}=b$$

Subsets and supersets

A subset of B ~ $A \subseteq B$

B is superset of A

$$\iff \big(\forall x.\ x \in A \Rightarrow x \in B\big)$$

$$A \subsetneq B \iff \big(A \subseteq B \wedge A \neq B\big)$$

<u>NB</u>:

$$A = B \iff \big[(A \subseteq B) \wedge (B \subseteq A)\big]$$

Lemma 103

1. *Reflexivity.*

   For all sets A, $A \subseteq A$.

2. *Transitivity.*

   For all sets A, B, C, $(A \subseteq B \ \wedge \ B \subseteq C) \implies A \subseteq C$.

3. *Antisymmetry.*

   For all sets A, B, $(A \subseteq B \ \wedge \ B \subseteq A) \implies A = B$.

Let A, B, C be sets.

Assume $A \subseteq B$ and $B \subseteq C$

$$\Updownarrow \qquad\qquad\qquad\qquad \Updownarrow$$

① $\forall x.\ x \in A \Rightarrow x \in B$ ② $\forall y.\ y \in B \Rightarrow y \in C$

RTP: $A \subseteq C$

$\Leftrightarrow \forall z.\ z \in A \Rightarrow z \in C$

Let z be arbitrary such that $z \in A$.

By ①, $z \in B$ and by ② $z \in C$.

$\boxtimes$

NB: $a \in \{x \in A \mid P(x)\} \iff [a \in A \wedge P(a)]$

Separation principle

For any set A and any definable property P, there is a set containing precisely those elements of A for which the property P holds.

$$\{ x \in A \mid \mathrm{P}(x) \}$$

Russell's paradox

If $\quad \mathcal{U} = \{x \mid x \notin x\}$ is a set

Then $\quad x \in \mathcal{U} \iff x \notin x \quad \forall x$

So $\quad \mathcal{U} \in \mathcal{U} \iff \mathcal{U} \notin \mathcal{U} \quad \lightning$

NB: $x \in \emptyset \iff \mathit{false}$

$\neg(x \in \emptyset)$

Empty set

Set theory has an

empty set ,

typically denoted

$\emptyset$ or $\{\}$,

with no elements.

NB: $\emptyset \subseteq A$

iff

$\forall x .\, x \in \emptyset \Rightarrow x \in A$

Cardinality

The *cardinality* of a set specifies its size. If this is a natural number, then the set is said to be *finite*.

Typical notations for the cardinality of a set S are $\#S$ or $|S|$.

Example:

$$\#\emptyset = 0$$

Finite sets

The *finite sets* are those with cardinality a natural number.

Example: For $n \in \mathbb{N}$,

$$[n] = \{ x \in \mathbb{N} \mid x < n \}$$

is finite of cardinality n.

$[2] = \{0, 1\}$

$\mathcal{P}([2]) = \{\emptyset, \{0,1\}, \{0\}, \{1\}\}$

$\mathcal{P}(\emptyset) = \{\emptyset\}$

Powerset axiom

For any set, there is a set consisting of all its subsets.

NB: $\emptyset \in \mathcal{P}(U)$

$U \in \mathcal{P}(U)$

$\#\emptyset = 0$

$\mathcal{P}(U)$

$\#\mathcal{P}(\emptyset) = 1$

$$\forall X.\ X \in \mathcal{P}(U) \iff X \subseteq U\ .$$

$\#\mathcal{P}([1]) = 2$

$\mathcal{P}([1]) = \{\emptyset, \{1\}\}$

$\#\mathcal{P}([2]) = 4$

$\#\mathcal{P}([n]) = 2^n$

NB: The powerset construction can be iterated. In particular,

$$\mathcal{F} \in \mathcal{P}\big(\mathcal{P}(U)\big) \iff \mathcal{F} \subseteq \mathcal{P}(U) \quad ;$$

that is, $\mathcal{F}$ is a set of subsets of U, sometimes referred to as a *family*.

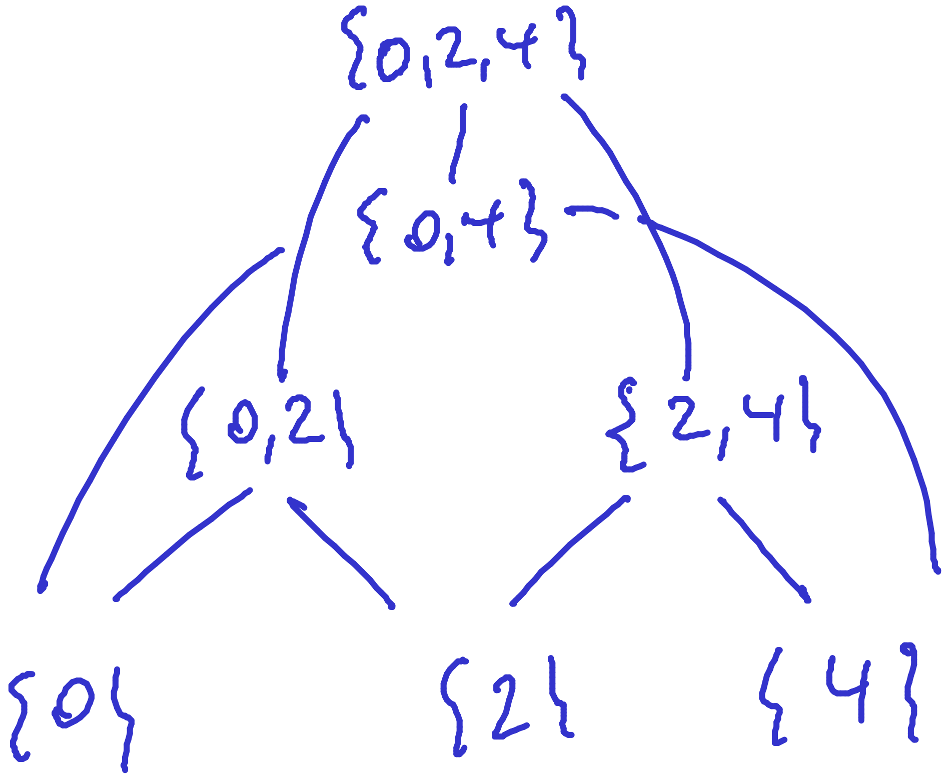
Example: The family $\mathcal{E} \subseteq \mathcal{P}\big(\,[5]\,\big)$ consisting of the non-empty subsets of $[5] = \{\,0,1,2,3,4\,\}$ whose elements are even is

$$\mathcal{E} \;=\; \big\{\,\{0\},\{2\},\{4\},\{0,2\},\{0,4\},\{2,4\},\{0,2,4\}\,\big\} \quad .$$

Hasse diagrams

indicate inclusion

$\mathcal{E}$



Proposition 104 *For all finite sets* $\mathcal{U}$,

$$\#\mathcal{P}(\mathcal{U}) = 2^{\#\mathcal{U}} \quad .$$

PROOF IDEA:

$\mathcal{U} = \{a_1, \ldots, a_n\}$ $\quad\quad \#\mathcal{U} = n$.

$\mathcal{P}(\mathcal{U}) = \{X \mid X \subseteq \mathcal{U}\}$

$X \subseteq \mathcal{U}$ determined by membership of each a_i.

To count $\mathcal{P}(\mathcal{U})$ is to count all the sequences of 0's & 1's of length $n = \#\mathcal{U}$

$$\begin{array}{cccc} 1 & 0 & 1 & \ldots \\ a_1 & a_2 & \ldots & a_n \end{array}$$

$a_1 \in X$, $a_2 \notin X$, $a_3 \in X$

So: $\#\mathcal{P}(\mathcal{U}) = 2^n$