Hoare logic and Model checking

Revision class

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Hoare logic and separation logic

Program variable assignment vs heap assignment

(Program variable) assignment
$X := E$ updates program variable X.

Heap assignment
$[E_1] := E_2$ (note the brackets) evaluates E_1 and, if E_1 evaluates to a pointer to an allocated heap location ℓ, writes to the heap at ℓ.

E.g. heap assignment $[X] := E$ (note the brackets) reads program variable X and, if the current value of X is a pointer to an allocated heap location ℓ, writes to the heap at ℓ, leaving X unchanged.

Whether to apply the rule for **(program variable) assignment** from lecture 1, or the separation logic rule for **heap assignment** depends on the command.

The concept of ownership

Ownership of a heap cell is the permission to safely read/write/dispose of it. **This ownership is not duplicable**.

E.g.: use-after-free: $\mathit{dispose}(X); [X] := 5$

Separation logic:

$\{X \mapsto v\}$
dispose(X);
$\{emp\}$
proof fails
$\{X \mapsto v\}$
[X] := 5
$\{X \mapsto 5\}$

If ownership was duplicable:

$\{X \mapsto v\}$
$\{X \mapsto v * X \mapsto v\}$
dispose(X);
$\{X \mapsto v\}$
[X] := 5
$\{X \mapsto 5\}$

How is ownership related to framing?

If we have proved $\{P\}\ C\ \{Q\}$ for some program C and we want to use this triple in a proof involving assertion R, we can use the frame rule to conclude $\{P * R\}\ C\ \{Q * R\}$: R is preserved by C.

$$\frac{\vdash \{P\}\ C\ \{Q\} \qquad mod(C) \cap FV(R) = \emptyset}{\vdash \{P * R\}\ C\ \{Q * R\}}$$

Intuitively: P must have all the ownership required for the safe execution of C — all the parts of the heap that C manipulates. The separating conjunction ensures that R cannot have ownership of those heap locations (or the precondition is false).

Recall: $P * R$ requires the disjointness of the heap cells for which P and R assert ownership.

Pure assertions

$$\begin{aligned}
[\![-]\!](=) &: Assertion \to Stack \to \mathcal{P}(Heap) \\
[\![\bot]\!](s) &\stackrel{def}{=} \emptyset \\
[\![\top]\!](s) &\stackrel{def}{=} Heap \\
[\![P \wedge Q]\!](s) &\stackrel{def}{=} [\![P]\!](s) \cap [\![Q]\!](s) \\
[\![P \vee Q]\!](s) &\stackrel{def}{=} [\![P]\!](s) \cup [\![Q]\!](s) \\
[\![P \Rightarrow Q]\!](s) &\stackrel{def}{=} \{h \in Heap \mid h \in [\![P]\!](s) \Rightarrow h \in [\![Q]\!](s)\} \\
&\vdots
\end{aligned}$$

What is the meaning of pure assertions, such as $\top$ or $t_1 = t_2$? Do they implicitly require the heap to be empty?

Semantics of pure assertions

$$[\![-]\!](=) : Assertion \to Stack \to \mathcal{P}(Heap)$$

$$[\![t_1 = t_2]\!](s) = \{h \mid [\![t_1]\!](s) = [\![t_2]\!](s)\} = \begin{cases} Heap & \text{if } [\![t_1]\!](s) = [\![t_2]\!](s) \\ \emptyset & \text{otherwise} \end{cases}$$

More generally, the semantics of a pure assertion in a stack s:

Informally: "check the pure assertion in s"; if it holds in s, return the set of all heaps, if not return the empty set of heaps.

Formally: don't worry about it, because we have not defined it.

Semantics of pure assertions, wrt. heap (continued). Fixed

The 2019 exam paper 8, question 7 asks:

$\{N = n \wedge N \geq 0\}$
X := null; while N > 0 do (X := alloc(N, X); N := N −1)
$\{\text{list}(X, [1, \ldots, n])\}$

(I have not checked whether that year used different definitions from ours, but) **This seems to be missing emp in the pre-condition:** $\{N = n \wedge N \geq 0 \wedge \text{emp}\}$

Why? $\{N = n \wedge N \geq 0\}$ makes no statement about the heap — the precondition is satisfied by any heap (and suitable stack). But without the emp requirement, we would not be able to prove the post-condition $\{\text{list}(X, [1, \ldots, n])\}$, which asserts that the **only** ownership is that of the list predicate instance.

Another error

Related: error in 2021 Paper 8 Question 8.

The pre-condition should have

$$\cdots \wedge 1 \leq S$$

instead of

$$\cdots * 1 \leq S$$

.

Conjunction and separating conjunction

What are the differences between them and when to use which?
And how do they interact with pure assertions?

$$[\![P * Q]\!](s) \stackrel{def}{=} \left\{ h \in Heap \;\middle|\; \exists h_1, h_2. \begin{array}{l} h_1 \in [\![P]\!](s) \wedge \\ h_2 \in [\![Q]\!](s) \wedge \\ h = h_1 \uplus h_2 \end{array} \right\}$$

$$[\![P \wedge Q]\!](s) \stackrel{def}{=} [\![P]\!](s) \cap [\![Q]\!](s)$$

Conjunction and separating conjunction (continued)

$$[\![P * Q]\!](s) \stackrel{def}{=} \left\{ h \in Heap \;\middle|\; \exists h_1, h_2. \begin{array}{l} h_1 \in [\![P]\!](s) \wedge \\ h_2 \in [\![Q]\!](s) \wedge \\ h = h_1 \uplus h_2 \end{array} \right\}$$

$$[\![P \wedge Q]\!](s) \stackrel{def}{=} [\![P]\!](s) \cap [\![Q]\!](s)$$

$p_1 \mapsto v_1 * p_2 \mapsto v_2$ **vs.** $p_1 \mapsto v_1 \wedge p_2 \mapsto v_2$

- $p_1 \mapsto v_1 * p_2 \mapsto v_2$ holds for a heap h that is the disjoint union of heaplets h_1 and h_2, where h_1 contains just cell p_1, with value v_1, and h_2 just cell p_2, with value v_2. So: ownership of **two disjoint** heap cells p_1 and p_2 with $p_1 \neq p_2$.
- $p_1 \mapsto v_1 \wedge p_2 \mapsto v_2$ holds for a heap h that satisfies two assertions simultaneously (is in the intersection of their interpretations):
(1) $p_1 \mapsto v_1$: h is a heap of just one heap cell, p_1 with value v_1
(2) $p_2 \mapsto v_2$: h is a heap of just one heap cell, p_2 with value v_2
So: ownership of just **one** heap cell, $p_1 = p_2$ with value $v_1 = v_2$.

Conjunction and separating conjunction (continued)

$$[\![P * Q]\!](s) \stackrel{def}{=} \left\{ h \in Heap \;\middle|\; \exists h_1, h_2. \begin{array}{l} h_1 \in [\![P]\!](s) \wedge \\ h_2 \in [\![Q]\!](s) \wedge \\ h = h_1 \uplus h_2 \end{array} \right\}$$

$$[\![P \wedge Q]\!](s) \stackrel{def}{=} [\![P]\!](s) \cap [\![Q]\!](s)$$

$(p \mapsto 1) * Y = 0$ **vs.** $(p \mapsto 1) \wedge Y = 0$

- $(p \mapsto 1) * Y = 0$ holds for a stack s and a heap h where h is the disjoint union of heaplets h_1 and h_2, such that h_1 contains ownership of one cell, p with value 1, and h_2 is an arbitrary heap if s satisfies $Y = 0$. So, s must map Y to 0 and h is the disjoint union of the heaplet of just p with value 1 and **an arbitrary disjoint** heap h_2.
- $(p \mapsto 1) \wedge Y = 0$ holds for a stack s and a heap h satisfying two assertion simultaneously: $p \mapsto 1$ and $Y = 0$. This means s must map Y to 0 and h must be the heap consisting of just that one cell.

It is good to be careful about the unexpected interaction of the usual logical connectives with the new separation logic connectives!

Example: 2019-p08-q07, e

Give a loop invariant for the following list concatenation triple:

$\{\mathrm{list}(X, \alpha) * \mathrm{list}(Y, \beta)\}$
if X = null then
  Z:=Y
else (
  Z := X; U := Z; V := [Z + 1];
  while V $\neq$ null do (U := V ; V := [V + 1]);
  [U + 1] := Y
)
$\{\mathrm{list}(Z, \alpha +\!\!+ \beta)\}$

Example: 2019-p08-q07, e

$\{\mathrm{list}(X, \alpha) * \mathrm{list}(Y, \beta)\}$
if X = null then

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  Z := X; U := Z; V := [Z + 1];
  while V $\neq$ null do (U := V ; V := [V + 1]);
  [U + 1] := Y

)
$\{\mathrm{list}(Z, \alpha +\!\!+ \beta)\}$

Example: 2019-p08-q07, e

$\{(\mathrm{list}(X, \alpha) * \mathrm{list}(Y, \beta)) \wedge X \neq \mathrm{null}\}$
Z := X; U := Z; V := [Z + 1];
while V $\neq$ null do (U := V ; V := [V + 1]);
[U + 1] := Y
$\{\mathrm{list}(Z, \alpha +\!\!+ \beta)\}$

$\{(\mathrm{list}(X,\alpha) * \mathrm{list}(Y,\beta)) \wedge X \neq \mathrm{null}\}$

$\{\exists t, p, \delta.\ \alpha = [t] \mathbin{+\!\!+} \delta \wedge (X \mapsto t, p * \mathrm{list}(p,\delta) * \mathit{list}(Y,\beta))\}$

Z := X;

$\{\exists t, p, \delta.\ \alpha = [t] \mathbin{+\!\!+} \delta \wedge (Z \mapsto t, p * \mathrm{list}(p,\delta) * \mathit{list}(Y,\beta))\}$

U := Z;

$\{\exists t, p, \delta.\ \alpha = [t] \mathbin{+\!\!+} \delta \wedge U = Z \wedge (Z \mapsto t, p * \mathrm{list}(p,\delta) * \mathit{list}(Y,\beta))\}$

V := [Z + 1];

$\{\exists t, \delta.\ \alpha = [t] \mathbin{+\!\!+} \delta \wedge U = Z \wedge (Z \mapsto t, V * \mathrm{list}(V,\delta) * \mathit{list}(Y,\beta))\}$

$I : \{\exists \gamma, t, \delta.\ \alpha = \gamma \mathbin{+\!\!+} [t] \mathbin{+\!\!+} \delta \wedge (\mathrm{plist}(Z,\gamma,U) * \mathrm{plist}(U,[t],V) * \mathrm{list}(V,\delta) * \mathit{list}(Y,\beta))\}$

while V $\neq$ null do (U := V ; V := [V + 1]);

$\{\exists \gamma, t, \delta.\ \alpha = \gamma \mathbin{+\!\!+} [t] \mathbin{+\!\!+} \delta \wedge (\mathrm{plist}(Z,\gamma,U) * \mathrm{plist}(U,[t],V) * \mathrm{list}(V,\delta) * \mathit{list}(Y,\beta))$

$\quad \wedge \neg(V \neq \mathrm{null})\}$

[U + 1] := Y

$\{\exists \gamma, t, \delta.\ \alpha = \gamma \mathbin{+\!\!+} [t] \mathbin{+\!\!+} \delta \wedge (\mathrm{plist}(Z,\gamma,U) * \mathrm{plist}(U,[t],Y) * \mathrm{list}(V,\delta) * \mathit{list}(Y,\beta))$

$\quad \wedge \neg(V \neq \mathrm{null})\}$

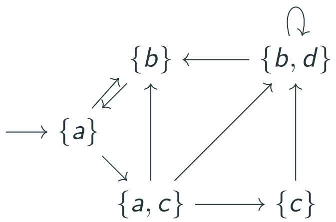
$\{\mathrm{list}(Z, \alpha \mathbin{+\!\!+} \beta)\}$

Model Checking

Temporal operators, e.g. in CTL

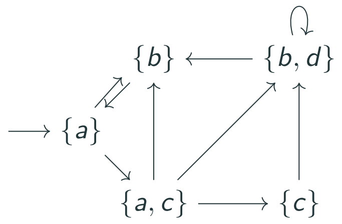
- $AX\psi$ and $EX\psi$:
  - Does the state satisfying ψ have to be different from the starting state?
  - Does ψ have to continue holding?
- $A(\psi_1 U \psi_2)$ and $E(\psi_1 U \psi_2)$:
  - Does ψ_1 have to continue holding?
  - What about ψ_2?

LTL examples



ϕ	$M \vDash \phi$
a	yes
Xa	no
Fb	yes
Fc	no
$(a \vee b)Uc$	no
dUa	yes
$G(a \vee b \vee c)$	yes
GFb	yes
FGb	no

CTL examples



ψ	$M \models \psi$
$EX(b \wedge \neg c)$	yes
AFd	no
EFd	yes
$E(aUd)$	yes
$AGEFd$	yes
$AFEGd$	no
$EFEGd$	yes
$E((a \vee c)U(EGb))$	yes

LTL/CTL expressivity

An elevator property: **"If it is possible to answer a call to some level in the next step, then the elevator does that"**
CTL: $\psi = \text{A G } ((\text{Call}_2 \wedge \text{E X Loc}_2) \rightarrow \text{A X Loc}_2)$

Q: Can we express the same in LTL with
$\phi = \text{G } (\text{Call}_2 \wedge (\text{Loc}_1 \vee \text{Loc}_3)) \rightarrow \text{X Loc}_2$?

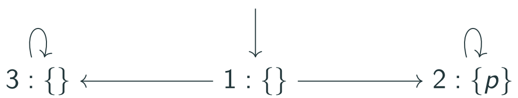
This depends on the details of the elevator temporal model.[1] In any case, ψ and ϕ are not generally equivalent. The point is: expressing properties of the tree of possible paths out of a given state — such as asserting the **existence** of some path — is not possible with LTL.

[1] I think — the way we have sketched the elevator in lecture 7 — this will not work: $\text{Loc}_1 \vee \text{Loc}_3$ does not imply there exists a next step such that Loc_2 holds.

LTL/CTL expressivity

An LTL formula not expressible in CTL: $\phi = (\text{F } p) \rightarrow (\text{F } q)$.

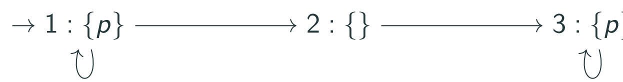
a) CTL formula $\psi_1 = (\text{A F } p) \rightarrow (\text{A F } q)$.
ϕ does not hold, ψ_1 does.



b) CTL formula $\psi_2 = \text{A G } (p \rightarrow (\text{A F } q))$.
ϕ holds, ψ_2 does not.

LTL/CTL expressivity

Why are F G p in LTL and A F A G p in CTL not equivalent?



Two kinds of infinite paths: (L1) loop in 1 forever, (L2) loop in 3 forever. Both kinds of paths **eventually** reach a state in which p holds **generally** (1 or 3, respectively). So F G p holds.

Informally: A F A G p holds if (check CTL (CTL*) semantics):

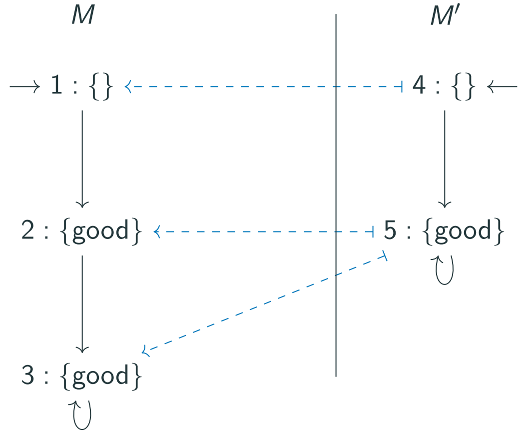
- all paths π from 1 satisfy F A G p, so
- all paths π from 1 eventually reach a state where A G p holds

But path kind (L1) does not: never leaves 1, and in 1, A G p is not satisfied, because there exists a path π_2 that goes to 2 from there.

It is good to be careful about the unexpected interaction of the temporal operators, with other temporal operators and with path quantifiers.

Why have simulation relations and not simulation functions?

$AP = AP' = \{\text{good}\}$



M simulates M'

Good luck!