Non-example of a ccc

The category **Mon** of monoids has a terminal object and binary products, but is <u>not</u> a ccc

because of the following bijections between sets, where 1 denotes a one-element set and the corresponding one-element monoid:

$$
\begin{aligned}
\mathscr{P}(X) &\cong \mathbf{Set}(X, \mathbb{Z}_2) \\
&\cong \mathbf{Mon}(\mathtt{List}\, X, \mathbb{Z}_2) \\
&\cong \mathbf{Mon}(1 \times \mathtt{List}\, X, \mathbb{Z}_2)
\end{aligned}
$$

by universal property of the free monoid $\mathtt{List}\, X$ on the set X

$X \cong 1 \times X$

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because of the following bijections between sets, where $\mathtt{1}$ denotes a one-element set and the corresponding one-element monoid:

$$\begin{aligned} \mathscr{P}(X) &\cong \mathbf{Set}(X, \mathbb{Z}_2) \\ &\cong \mathbf{Mon}(\mathtt{List}\, X, \mathbb{Z}_2) \\ &\cong \mathbf{Mon}(\mathtt{1} \times \mathtt{List}\, X, \mathbb{Z}_2) \end{aligned}$$

Since the one-element monoid is initial in **Mon**, for any $M \in \mathbf{Mon}$, we have $\mathbf{Mon}(\mathtt{1}, M) \cong \mathtt{1}$ and hence

$$\mathtt{List}\, X \Rightarrow \mathbb{Z}_2 \text{ exists in } \mathbf{Mon} \text{ iff } \mathscr{P}(X) \cong \mathtt{1} \text{ iff } X = 0$$

Btw, a ccc has a zero object if, and only if, it is trivial (check).

Cartesian closed pre-order

Recall that each preorder $\underline{P} = (P, \sqsubseteq)$ gives a category $\mathbf{C}_{\underline{P}}$. It is a biccc iff $\underline{P}$ has

- a greatest element $\top$: $\forall p \in P,\ p \sqsubseteq \top$
- a least element $\bot$: $\forall p \in P,\ \bot \sqsubseteq p$
- binary meets $p \wedge q$:
  $\forall r \in P,\ r \sqsubseteq p \wedge q \ \Leftrightarrow\ r \sqsubseteq p \wedge r \sqsubseteq q$
- binary joins $p \vee q$:
  $\forall r \in P,\ p \vee q \sqsubseteq r \ \Leftrightarrow\ p \sqsubseteq r \wedge q \sqsubseteq r$
- Heyting implications $p \rightarrow q$:
  $\forall r \in P,\ r \sqsubseteq p \rightarrow q \ \Leftrightarrow\ r \wedge p \sqsubseteq q$

Examples:

- Any Boolean algebra (with $p \to q = \neg p \vee q$).
- $([0,1], \leq)$ with $\top = 1$, $\bot = 0$, $p \wedge q = \min\{p, q\}$, $p \vee q = \max\{p, q\}$, and $p \to q = \begin{cases} 1 & \text{if } p \leq q \\ q & \text{if } q < p \end{cases}$

Intuitionistic Propositional Logic (IPL)

We present it in "natural deduction" style and only consider the fragment with conjunction and implication, with the following syntax:

Formulas of IPL: $\varphi, \psi, \theta, \ldots ::=$

$p, q, r, \ldots$	propositional identifiers
$\mathtt{true}$	truth
$\varphi \mathbin{\&} \psi$	conjunction
$\varphi \Rightarrow \psi$	implication

Sequents of IPL: $\Phi ::= \diamond$ empty

Φ, φ non-empty

(so sequents are finite lists of formulas)

IPL entailment $\Phi \vdash \varphi$

The intended meaning of $\Phi \vdash \varphi$ is "the conjunction of the formulas in Φ implies the formula φ". The relation $\_ \vdash \_$ is inductively generated by the following rules:

$$\frac{}{\Phi, \varphi \vdash \varphi}\ (\textsc{ax}) \qquad \frac{\Phi \vdash \varphi}{\Phi, \psi \vdash \varphi}\ (\textsc{wk}) \qquad \frac{\Phi \vdash \varphi \qquad \Phi, \varphi \vdash \psi}{\Phi \vdash \psi}\ (\textsc{cut})$$

$$\frac{}{\Phi \vdash \mathtt{true}}\ (\textsc{true}) \qquad \frac{\Phi \vdash \varphi \qquad \Phi \vdash \psi}{\Phi \vdash \varphi \mathbin{\&} \psi}\ (\&\textsc{i}) \qquad \frac{\Phi, \varphi \vdash \psi}{\Phi \vdash \varphi \Rightarrow \psi}\ ({\Rightarrow}\textsc{i})$$

$$\frac{\Phi \vdash \varphi \mathbin{\&} \psi}{\Phi \vdash \varphi}\ (\&\textsc{e}_1) \qquad \frac{\Phi \vdash \varphi \mathbin{\&} \psi}{\Phi \vdash \psi}\ (\&\textsc{e}_2) \qquad \frac{\Phi \vdash \varphi \Rightarrow \psi \qquad \Phi \vdash \varphi}{\Phi \vdash \psi}\ ({\Rightarrow}\textsc{e})$$

For example, if $\Phi = \diamond, \varphi \Rightarrow \psi, \psi \Rightarrow \theta$, then $\Phi \vdash \varphi \Rightarrow \theta$ is provable in IPL, because:

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{}{\Phi \vdash \psi \Rightarrow \theta}\,(\textsc{ax})
    }{\Phi, \varphi \vdash \psi \Rightarrow \theta}\,(\textsc{wk})
    \qquad
    \dfrac{
      \dfrac{
        \dfrac{
          \dfrac{}{\diamond, \varphi \Rightarrow \psi \vdash \varphi \Rightarrow \psi}\,(\textsc{ax})
        }{\Phi \vdash \varphi \Rightarrow \psi}\,(\textsc{wk})
      }{\Phi, \varphi \vdash \varphi \Rightarrow \psi}\,(\textsc{wk})
      \qquad
      \dfrac{}{\Phi, \varphi \vdash \varphi}\,(\textsc{ax})
    }{\Phi, \varphi \vdash \psi}\,(\Rightarrow\textsc{e})
  }{\Phi, \varphi \vdash \theta}\,(\Rightarrow\textsc{e})
}{\Phi \vdash \varphi \Rightarrow \theta}\,(\Rightarrow\textsc{i})
$$

Semantics of IPL

in a cartesian closed pre-order $(P, \sqsubseteq)$

Given a function M assigning a meaning to each propositional identifier p as an element $M(p) \in P$, we can assign meanings to IPL formula φ and sequents Φ as elements $M[\![\varphi]\!], M[\![\Phi]\!] \in P$ by recursion on their structure:

$M[\![p]\!] = M(p)$	
$M[\![\mathtt{true}]\!] = \top$	greatest element
$M[\![\varphi \mathbin{\&} \psi]\!] = M[\![\varphi]\!] \wedge M[\![\psi]\!]$	binary meet
$M[\![\varphi \Rightarrow \psi]\!] = M[\![\varphi]\!] \to M[\![\psi]\!]$	Heyting implication
$M[\![\diamond]\!] = \top$	greatest element
$M[\![\Phi, \varphi]\!] = M[\![\Phi]\!] \wedge M[\![\varphi]\!]$	binary meet

Semantics of IPL

in a cartesian closed pre-order $(P, \sqsubseteq)$

Soundness Theorem. If $\Phi \vdash \varphi$ is provable from the rules of IPL, then $M[\![\Phi]\!] \sqsubseteq M[\![\varphi]\!]$ holds in any cartesian closed pre-order.

Proof. exercise (show that $\{(\Phi, \varphi) \mid M[\![\Phi]\!] \sqsubseteq M[\![\varphi]\!]\}$ is closed under the rules defining IPL entailment and hence contains $\{(\Phi, \varphi) \mid \Phi \vdash \varphi\}$)

Example

Peirce's Law $\diamond \vdash ((\varphi \Rightarrow \psi) \Rightarrow \varphi) \Rightarrow \varphi$
is <u>not</u> provable in IPL.

(whereas the formula $((\varphi \Rightarrow \psi) \Rightarrow \varphi) \Rightarrow \varphi$ is a classical tautology)

Example

Peirce's Law $\diamond \vdash ((\varphi \Rightarrow \psi) \Rightarrow \varphi) \Rightarrow \varphi$

is <u>not</u> provable in IPL.

(whereas the formula $((\varphi \Rightarrow \psi) \Rightarrow \varphi) \Rightarrow \varphi$ is a classical tautology)

For if $\diamond \vdash ((\varphi \Rightarrow \psi) \Rightarrow \varphi) \Rightarrow \varphi$ were provable in IPL, then by the Soundness Theorem we would have
$\top = M[\![\diamond]\!] \sqsubseteq M[\![((\varphi \Rightarrow \psi) \Rightarrow \varphi) \Rightarrow \varphi]\!]$.

But in the cartesian closed poset $([0, 1], \leq)$, taking $M(p) = 1/2$ and $M(q) = 0$, we get

$$\begin{aligned} M[\![((p \Rightarrow q) \Rightarrow p) \Rightarrow p]\!] &= ((1/2 \rightarrow 0) \rightarrow 1/2) \rightarrow 1/2 \\ &= (0 \rightarrow 1/2) \rightarrow 1/2 \\ &= 1 \rightarrow 1/2 \\ &= 1/2 \\ &\not\geq 1 \end{aligned}$$

Semantics of IPL

in a cartesian closed preorder $(P, \sqsubseteq)$

Completeness Theorem. Given Φ, φ, if for all cartesian closed preorders $(P, \sqsubseteq)$ and all interpretations M of the propositional identifiers as elements of P, it is the case that $M[\![\Phi]\!] \sqsubseteq M[\![\varphi]\!]$ holds in P, then $\Phi \vdash \varphi$ is provable in IPL.

Semantics of IPL

in a cartesian closed preorder $(P, \sqsubseteq)$

Completeness Theorem. Given Φ, φ, if for all cartesian closed preorders $(P, \sqsubseteq)$ and all interpretations M of the propositional identifiers as elements of P, it is the case that $M[\![\Phi]\!] \sqsubseteq M[\![\varphi]\!]$ holds in P, then $\Phi \vdash \varphi$ is provable in IPL.

Proof. Define

$$
\begin{aligned}
P &\triangleq \{\text{formulas of IPL}\} \\
\varphi \sqsubseteq \psi &\triangleq \diamond, \varphi \vdash \psi \text{ is provable in IPL}
\end{aligned}
$$

Then one can show that $(P, \sqsubseteq)$ is a cartesian closed preorder.
For this preorder, taking M to be $M(p) = p$, one can show that $M[\![\Phi]\!] \sqsubseteq M[\![\varphi]\!]$ holds in P iff $\Phi \vdash \varphi$ is provable in IPL. □

Proof theory

Two IPL proofs of $\diamond, \varphi \Rightarrow \psi, \psi \Rightarrow \theta \vdash \varphi \Rightarrow \theta$

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{}{\cdots}\,(\textsc{ax})
    }{
      \Phi, \varphi \vdash \psi \Rightarrow \theta
    }\,(\textsc{wk})
    \qquad
    \dfrac{
      \dfrac{
        \dfrac{\dfrac{}{\cdots}\,(\textsc{ax})}{\cdots}\,(\textsc{wk})
      }{
        \Phi, \varphi \vdash \varphi \Rightarrow \psi
      }\,(\textsc{wk})
      \qquad
      \dfrac{}{\Phi, \varphi \vdash \varphi}\,(\textsc{ax})
    }{
      \Phi, \varphi \vdash \psi
    }\,(\Rightarrow\textsc{e})
  }{
    \Phi, \varphi \vdash \theta
  }\,(\Rightarrow\textsc{e})
}{
  \Phi \vdash \varphi \Rightarrow \theta
}\,(\Rightarrow\textsc{i})
$$

where $\Phi \triangleq \diamond, \varphi \Rightarrow \psi, \psi \Rightarrow \theta$

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{\dfrac{}{\cdots}\,(\textsc{ax})}{\cdots}\,(\textsc{wk})
      }{
        \Psi \vdash \varphi \Rightarrow \psi
      }\,(\textsc{wk})
      \qquad
      \dfrac{}{\Psi \vdash \varphi}\,(\textsc{ax})
    }{
      \Psi \vdash \psi
    }\,(\Rightarrow\textsc{e})
    \qquad
    \dfrac{
      \dfrac{
        \dfrac{\dfrac{}{\cdots}\,(\textsc{ax})}{\cdots}\,(\textsc{wk})
      }{
        \Psi, \psi \vdash \psi \Rightarrow \theta
      }\,(\textsc{wk})
      \qquad
      \dfrac{}{\Psi, \psi \vdash \psi}\,(\textsc{ax})
    }{
      \Psi, \psi \vdash \theta
    }\,(\Rightarrow\textsc{e})
  }{
    \Psi \vdash \theta
  }\,(\textsc{cut})
}{
  \diamond, \varphi \Rightarrow \psi, \psi \Rightarrow \theta \vdash \varphi \Rightarrow \theta
}\,(\Rightarrow\textsc{i})
$$

where $\Psi \triangleq \diamond, \varphi \Rightarrow \psi, \psi \Rightarrow \theta, \varphi$

Proof theory

Two IPL proofs of $\diamond, \varphi \Rightarrow \psi, \psi \Rightarrow \theta \vdash \varphi \Rightarrow \theta$

$$
\dfrac{
  \dfrac{\dfrac{}{\cdots}\,(\textsc{ax})}{\Phi, \varphi \vdash \psi \Rightarrow \theta}\,(\textsc{wk})
  \qquad
  \dfrac{
    \dfrac{\dfrac{\dfrac{}{\cdots}\,(\textsc{ax})}{\cdots}\,(\textsc{wk})}{\Phi, \varphi \vdash \varphi \Rightarrow \psi}\,(\textsc{wk})
    \qquad
    \dfrac{}{\Phi, \varphi \vdash \varphi}\,(\textsc{ax})
  }{\Phi, \varphi \vdash \psi}\,({\Rightarrow}\textsc{e})
}{
  \dfrac{\Phi, \varphi \vdash \theta}{\Phi \vdash \varphi \Rightarrow \theta}\,({\Rightarrow}\textsc{i})
}\,({\Rightarrow}\textsc{e})
$$

where $\Phi \triangleq \diamond, \varphi \Rightarrow \psi, \psi \Rightarrow \theta$

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{\dfrac{\dfrac{}{\cdots}\,(\textsc{ax})}{\cdots}\,(\textsc{wk})}{\Psi \vdash \varphi \Rightarrow \psi}\,(\textsc{wk})
      \qquad
      \dfrac{}{\Psi \vdash \varphi}\,(\textsc{ax})
    }{\Psi \vdash \psi}\,({\Rightarrow}\textsc{e})
    \qquad
    \dfrac{
      \dfrac{\dfrac{\dfrac{}{\cdots}\,(\textsc{ax})}{\cdots}\,(\textsc{wk})}{\Psi, \psi \vdash \psi \Rightarrow \theta}\,(\textsc{wk})
      \qquad
      \dfrac{}{\Psi, \psi \vdash \psi}\,(\textsc{ax})
    }{\Psi, \psi \vdash \theta}\,({\Rightarrow}\textsc{e})
  }{\Psi \vdash \theta}\,(\textsc{cut})
}{\diamond, \varphi \Rightarrow \psi, \psi \Rightarrow \theta \vdash \varphi \Rightarrow \theta}\,({\Rightarrow}\textsc{i})
$$

where $\Psi \triangleq \diamond, \varphi \Rightarrow \psi, \psi \Rightarrow \theta, \varphi$

Why is the first proof simpler than the second one?

Proof theory

$$\frac{}{\Phi, \varphi \vdash \varphi}\ (\textsc{ax}) \qquad \frac{\Phi \vdash \varphi}{\Phi, \psi \vdash \varphi}\ (\textsc{wk}) \qquad \frac{\Phi \vdash \varphi \qquad \Phi, \varphi \vdash \psi}{\Phi \vdash \psi}\ (\textsc{cut})$$

$$\frac{}{\Phi \vdash \mathtt{true}}\ (\textsc{true}) \qquad \frac{\Phi \vdash \varphi \qquad \Phi \vdash \psi}{\Phi \vdash \varphi \mathbin{\&} \psi}\ (\&\textsc{i}) \qquad \frac{\Phi, \varphi \vdash \psi}{\Phi \vdash \varphi \Rightarrow \psi}\ (\Rightarrow\textsc{i})$$

$$\frac{\Phi \vdash \varphi \mathbin{\&} \psi}{\Phi \vdash \varphi}\ (\&\textsc{e}_1) \qquad \frac{\Phi \vdash \varphi \mathbin{\&} \psi}{\Phi \vdash \psi}\ (\&\textsc{e}_2) \qquad \frac{\Phi \vdash \varphi \Rightarrow \psi \qquad \Phi \vdash \varphi}{\Phi \vdash \psi}\ (\Rightarrow\textsc{e})$$

FACT: if an IPL sequent $\Phi \vdash \phi$ is provable from the rules, it is provable without using the (cut) rule.

Proof theory

$$\frac{}{\Phi, \varphi \vdash \varphi}\ (\textsc{ax}) \qquad \frac{\Phi \vdash \varphi}{\Phi, \psi \vdash \varphi}\ (\textsc{wk}) \qquad \frac{\Phi \vdash \varphi \qquad \Phi, \varphi \vdash \psi}{\Phi \vdash \psi}\ (\textsc{cut})$$

$$\frac{}{\Phi \vdash \mathtt{true}}\ (\textsc{true}) \qquad \frac{\Phi \vdash \varphi \qquad \Phi \vdash \psi}{\Phi \vdash \varphi \,\&\, \psi}\ (\&\textsc{i}) \qquad \frac{\Phi, \varphi \vdash \psi}{\Phi \vdash \varphi \Rightarrow \psi}\ ({\Rightarrow}\textsc{i})$$

$$\frac{\Phi \vdash \varphi \,\&\, \psi}{\Phi \vdash \varphi}\ (\&\textsc{e}_1) \qquad \frac{\Phi \vdash \varphi \,\&\, \psi}{\Phi \vdash \psi}\ (\&\textsc{e}_2) \qquad \frac{\Phi \vdash \varphi \Rightarrow \psi \qquad \Phi \vdash \varphi}{\Phi \vdash \psi}\ ({\Rightarrow}\textsc{e})$$

FACT: if an IPL sequent $\Phi \vdash \phi$ is provable from the rules, it is provable without using the (cut) rule.

Simply-Typed Lambda Calculus provides a language for describing proofs in IPL and their properties.

Simply-Typed Lambda Calculus (STLC)

Types: $A, B, C, \ldots ::=$

$G, G', G'' \ldots$	"ground" types
$\mathtt{unit}$	unit type
$A \times B$	product type
$A \to B$	function type

Simply-Typed Lambda Calculus (STLC)

Types: $A, B, C, \ldots ::=$

$G, G', G'' \ldots$	"ground" types
$\mathtt{unit}$	unit type
$A \times B$	product type
$A \to B$	function type

Terms: $s, t, r, \ldots ::=$

c^A	constants (of given type A)
x	variable (countably many)
$()$	unit value
$(s\ ,\ t)$	pair
$\mathtt{fst}\ t \quad \mathtt{snd}\ t$	projections
$\lambda x : A.\, t$	function abstraction
$s\, t$	function application

STLC

Some examples of terms:

- $\lambda z : (A \to B) \times (A \to C).\, \lambda x : A.\, ((\mathtt{fst}\, z)\, x\, ,\, (\mathtt{snd}\, z)\, x))$

  (has type $((A \to B) \times (A \to C)) \to (A \to (B \times C))$)

- $\lambda z : A \to (B \times C).\, (\lambda x : A.\ \mathtt{fst}(z\, x)\, ,\, \lambda y : A.\ \mathtt{snd}(z\, y))$

  (has type $(A \to (B \times C)) \to ((A \to B) \times (A \to C))$)

- $\lambda z : A \to (B \times C).\, \lambda x : A.\, ((\mathtt{fst}\, z)\, x\, ,\, (\mathtt{snd}\, z)\, x)$

  (has no type)

STLC typing relation, $\Gamma \vdash t : A$

Γ ranges over typing environments

$$\Gamma ::= \diamond \mid \Gamma, x : A$$

(so typing environments are comma-separated lists of (variable,type)-pairs — in fact only the lists whose variables are mutually distinct get used)

The typing relation $\Gamma \vdash t : A$ is inductively defined by the following rules, which make use of the notation below

$\boxed{\Gamma\ \mathtt{ok}}$ means: no variable occurs more than once in Γ

$\boxed{\mathtt{dom}\,\Gamma}$ = finite set of variables occurring in Γ

STLC typing relation, $\Gamma \vdash t : A$

Typing rules for variables

$$\frac{\Gamma\ \mathtt{ok} \qquad x \notin \mathtt{dom}\,\Gamma}{\Gamma, x : A \vdash x : A}\ (\textsc{var})$$

$$\frac{\Gamma \vdash x : A \qquad x' \notin \mathtt{dom}\,\Gamma}{\Gamma, x' : A' \vdash x : A}\ (\textsc{var'})$$

Typing rules for constants and unit value

$$\frac{\Gamma\ \mathtt{ok}}{\Gamma \vdash c^{A} : A}\ (\textsc{cons})$$

$$\frac{\Gamma\ \mathtt{ok}}{\Gamma \vdash () : \mathtt{unit}}\ (\textsc{unit})$$

STLC typing relation, $\Gamma \vdash t : A$

Typing rules for pairs and projections

$$\frac{\Gamma \vdash s : A \qquad \Gamma \vdash t : B}{\Gamma \vdash (s\,,t) : A \times B}\ (\textsc{pair})$$

$$\frac{\Gamma \vdash t : A \times B}{\Gamma \vdash \texttt{fst}\ t : A}\ (\textsc{fst})$$

$$\frac{\Gamma \vdash t : A \times B}{\Gamma \vdash \texttt{snd}\ t : B}\ (\textsc{snd})$$

STLC typing relation, $\Gamma \vdash t : A$

Typing rules for function abstraction & application

$$\frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x : A.\, t : A \to B}\ (\textsc{fun})$$

$$\frac{\Gamma \vdash s : A \to B \qquad \Gamma \vdash t : A}{\Gamma \vdash s\, t : B}\ (\textsc{app})$$

STLC typing relation, $\Gamma \vdash t : A$

Example typing derivation:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash g : B \to C}\,(\text{VAR})}{\Gamma, x : A \vdash g : B \to C}\,(\text{VAR'}) \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\diamond, f : A \to B \vdash f : A \to B}\,(\text{VAR})}{\Gamma \vdash f : A \to B}\,(\text{VAR'})}{\Gamma, x : A \vdash f : A \to B}\,(\text{VAR'}) \quad \dfrac{}{\Gamma, x : A \vdash x : A}\,(\text{VAR})}{\Gamma, x : A \vdash f\,x : B}\,(\text{APP})}{\Gamma, x : A \vdash g(f\,x) : C}\,(\text{APP})}{\Gamma \vdash \lambda x : A.\, g(f\,x) : A \to C}\,(\text{FUN})}{\diamond, f : A \to B \vdash \lambda g : B \to C.\, \lambda x : A.\, g(f\,x) : (B \to C) \to (A \to C)}\,(\text{FUN})}{\diamond \vdash \lambda f : A \to B.\, \lambda g : B \to C.\, \lambda x : A.\, g(f\,x) : (A \to B) \to (B \to C) \to (A \to C)}\,(\text{FUN})
$$

where $\Gamma \triangleq \diamond, f : A \to B, g : B \to C$

NB: The STLC typing rules are "syntax-directed", by the structure of terms t and then in the case of variables x, by the structure of typing environments Γ.

Semantics of STLC types in a ccc

Given a cartesian closed category $\mathbf{C}$, any function M mapping ground types G to objects $M(G) \in \mathbf{C}$ extends to a function $A \mapsto M[\![A]\!] \in \mathbf{C}$ and $\Gamma \mapsto M[\![\Gamma]\!] \in \mathbf{C}$ from STLC types and typing environments to $\mathbf{C}$-objects, by recursion on their structure:

$M[\![G]\!] = M(G)$	an object in $\mathbf{C}$
$M[\![\mathtt{unit}]\!] = 1$	terminal object in $\mathbf{C}$
$M[\![A \times B]\!] = M[\![A]\!] \times M[\![B]\!]$	product in $\mathbf{C}$
$M[\![A \to B]\!] = M[\![A]\!] \Rightarrow M[\![B]\!]$	exponential in $\mathbf{C}$
$M[\![\diamond]\!] = 1$	terminal object in $\mathbf{C}$
$M[\![\Gamma, x : A]\!] = M[\![\Gamma]\!] \times M[\![A]\!]$	product in $\mathbf{C}$

Semantics of STLC terms in a ccc

Given a cartesian closed category $\mathbf{C}$, and

given any function M mapping

- ground types G to $\mathbf{C}$-objects $M(G)$
  (which extends to a function mapping all types to objects, $A \mapsto M[\![A]\!]$, as we have seen)

Semantics of STLC terms in a ccc

Given a cartesian closed category $\mathbf{C}$, and

given any function M mapping

- ground types G to $\mathbf{C}$-objects $M(G)$
- constants c^A to $\mathbf{C}$-morphisms $M(c^A) : 1 \to M[\![A]\!]$
  (In a category with a terminal object 1, given an object $X \in \mathbf{C}$, morphisms $1 \to X$ are typically called global elements of X.)

Semantics of STLC terms in a ccc

Given a cartesian closed category $\mathbf{C}$, and

given any function M mapping

- ground types G to $\mathbf{C}$-objects $M(G)$
- constants c^A to $\mathbf{C}$-morphisms $M(c^A) : 1 \to M[\![A]\!]$

we get a function mapping provable instances of the typing relation $\Gamma \vdash t : A$ to $\mathbf{C}$-morphisms

$$\boxed{M[\![\Gamma \vdash t : A]\!] : M[\![\Gamma]\!] \to M[\![A]\!]}$$

defined by recursing over the proof of $\Gamma \vdash t : A$ from the typing rules (which follows the structure of t):

Semantics of STLC terms in a ccc

Variables:

$$M[\![\Gamma, x : A \vdash x : A]\!] \;=\; M[\![\Gamma]\!] \times M[\![A]\!] \xrightarrow{\pi_2} M[\![A]\!]$$

$$M[\![\Gamma, x' : A' \vdash x : A]\!]$$
$$=\; M[\![\Gamma]\!] \times M[\![A']\!] \xrightarrow{\pi_1} M[\![\Gamma]\!] \xrightarrow{M[\![\Gamma \vdash x:A]\!]} M[\![A]\!]$$

Constants:

$$M[\![\Gamma \vdash c^A : A]\!] \;=\; M[\![\Gamma]\!] \xrightarrow{\langle\rangle} 1 \xrightarrow{M(c^A)} M[\![A]\!]$$

Unit value:

$$M[\![\Gamma \vdash () : \texttt{unit}]\!] \;=\; M[\![\Gamma]\!] \xrightarrow{\langle\rangle} 1$$

Semantics of STLC terms in a ccc

Pairing:

$$M[\![\Gamma \vdash (s\,,\,t) : A \times B]\!]$$
$$= \; M[\![\Gamma]\!] \xrightarrow{\langle M[\![\Gamma \vdash s : A]\!], M[\![\Gamma \vdash t : B]\!] \rangle} M[\![A]\!] \times M[\![B]\!]$$

Projections:

$$M[\![\Gamma \vdash \texttt{fst}\, t : A]\!]$$
$$= \; M[\![\Gamma]\!] \xrightarrow{M[\![\Gamma \vdash t : A \times B]\!]} M[\![A]\!] \times M[\![B]\!] \xrightarrow{\pi_1} M[\![A]\!]$$

Semantics of STLC terms in a ccc

Pairing:

$$\mathcal{M}[\![\Gamma \vdash (s\,,\,t) : A \times B]\!] = \mathcal{M}[\![\Gamma]\!] \xrightarrow{\langle \mathcal{M}[\![\Gamma \vdash s : A]\!], \mathcal{M}[\![\Gamma \vdash t : B]\!] \rangle} \mathcal{M}[\![A]\!] \times \mathcal{M}[\![B]\!]$$

Projections:

Given that $\Gamma \vdash \mathtt{fst}\, t : A$ holds, there is a unique type B such that $\Gamma \vdash t : A \times B$ already holds.

$$\mathcal{M}[\![\Gamma \vdash \mathtt{fst}\, t : A]\!] = \mathcal{M}[\![\Gamma]\!] \xrightarrow{\mathcal{M}[\![\Gamma \vdash t : A \times B]\!]} \mathcal{M}[\![A]\!] \times \mathcal{M}[\![B]\!] \xrightarrow{\pi_1} \mathcal{M}[\![A]\!]$$

Lemma. If $\Gamma \vdash t : A$ and $\Gamma \vdash t : B$ are provable, then $A = B$.

Semantics of STLC terms in a ccc

Pairing:

$$M[\![\Gamma \vdash (s\,,t) : A \times B]\!]$$

$$= \; M[\![\Gamma]\!] \xrightarrow{\langle M[\![\Gamma \vdash s : A]\!], M[\![\Gamma \vdash t : B]\!]\rangle} M[\![A]\!] \times M[\![B]\!]$$

Projections:

$$M[\![\Gamma \vdash \mathtt{snd}\, t : B]\!] =$$

$$M[\![\Gamma]\!] \xrightarrow{M[\![\Gamma \vdash t : A \times B]\!]} M[\![A]\!] \times M[\![B]\!] \xrightarrow{\pi_2} M[\![B]\!]$$

(As for the case of $\mathtt{fst}$, if $\Gamma \vdash \mathtt{snd}\, t : B$, then $\Gamma \vdash t : A \times B$ already holds for a unique type A.)

Semantics of STLC terms in a ccc

Function abstraction:

$$M[\![\Gamma \vdash \lambda x : A.t : A \to B]\!] \\ = \texttt{cur}\, f : M[\![\Gamma]\!] \longrightarrow (M[\![A]\!] \Rightarrow M[\![B]\!])$$

where

$$f = M[\![\Gamma, x : A \vdash t : B]\!] : M[\![\Gamma]\!] \times M[\![A]\!] \longrightarrow M[\![B]\!]$$

Semantics of STLC terms in a ccc

Function application:

$$M[\![\Gamma \vdash s\,t : B]\!]$$
$$= \; M[\![\Gamma]\!] \xrightarrow{\langle f,g \rangle} (M[\![A]\!] \Rightarrow M[\![B]\!]) \times M[\![A]\!] \xrightarrow{\mathtt{app}} M[\![B]\!]$$

where

$A \;=\;$ unique type such that $\Gamma \vdash s : A \to B$ and $\Gamma \vdash t : A$ already holds (exists because $\Gamma \vdash s\,t : B$ holds)

$f \;=\; M[\![\Gamma \vdash s : A \to B]\!] : M[\![\Gamma]\!] \longrightarrow (M[\![A]\!] \Rightarrow M[\![B]\!])$

$g \;=\; M[\![\Gamma \vdash t : A]\!] : M[\![\Gamma]\!] \longrightarrow M[\![A]\!]$

Example

Consider $t \triangleq \lambda x : A.\, g(f\, x)$ so that $\Gamma \vdash t : A \to C$ for $\Gamma \triangleq \diamond, f : A \to B, g : B \to C$.

Suppose $M[\![A]\!] = X$, $M[\![B]\!] = Y$ and $M[\![C]\!] = Z$ in $\mathbf{C}$. Then

$$M[\![\Gamma]\!] = (1 \times Y^X) \times Z^Y$$
$$M[\![\Gamma, x : A]\!] = ((1 \times Y^X) \times Z^Y) \times X$$

$$M[\![\Gamma, x : A \vdash x : A]\!] = \pi_2$$
$$M[\![\Gamma, x : A \vdash g : B \to C]\!] = \pi_2 \circ \pi_1$$
$$M[\![\Gamma, x : A \vdash f : A \to B]\!] = \pi_2 \circ \pi_1 \circ \pi_1$$
$$M[\![\Gamma, x : A \vdash f\, x : B]\!] = \mathtt{app} \circ \langle \pi_2 \circ \pi_1 \circ \pi_1 , \pi_2 \rangle$$
$$M[\![\Gamma, x : A \vdash g\,(f\, x) : C]\!] = \mathtt{app} \circ \langle \pi_2 \circ \pi_1 , \mathtt{app} \circ \langle \pi_2 \circ \pi_1 \circ \pi_1 , \pi_2 \rangle \rangle$$
$$M[\![\Gamma \vdash t : A \to C]\!] = \mathtt{cur}(\mathtt{app} \circ \langle \pi_2 \circ \pi_1 , \mathtt{app} \circ \langle \pi_2 \circ \pi_1 \circ \pi_1 , \pi_2 \rangle \rangle)$$