STLC equations

take the form $\boxed{\Gamma \vdash s = t : A}$ where $\Gamma \vdash s : A$ and $\Gamma \vdash t : A$ are provable.

Such an equation is satisfied by the semantics in a ccc if $M[\![\Gamma \vdash s : A]\!]$ and $M[\![\Gamma \vdash t : A]\!]$ are equal $\mathbf{C}$-morphisms $M[\![\Gamma]\!] \to M[\![A]\!]$.

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Ans: $(\alpha)\beta\eta$-equivalence — to define this, first have to define alpha-equivalence, substitution and its semantics.

Alpha equivalence of STLC terms

The names of λ-bound variables should not affect meaning.

E.g. $\lambda f : A \to B.\, \lambda x : A.\, f\, x$ should have the same meaning as $\lambda x : A \to B.\, \lambda f : A.\, x\, f$.

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This issue is best dealt with at the level of syntax rather than semantics: from now on we re-define "STLC term" to mean not an abstract syntax tree (generated as described before), but rather an equivalence class of such trees with respect to alpha-equivalence $\boxed{s =_\alpha t}$, defined as follows …

(Alternatively, one can use a "nameless" (de Bruijn) representation of terms.)

Alpha equivalence of STLC terms

$$\frac{}{c^A =_\alpha c^A} \qquad \frac{}{x =_\alpha x} \qquad \frac{}{() =_\alpha ()} \qquad \frac{s =_\alpha s' \qquad t =_\alpha t'}{(s\,,t) =_\alpha (s'\,,t')} \qquad \frac{t =_\alpha t'}{\mathtt{fst}\,t =_\alpha \mathtt{fst}\,t'}$$

$$\frac{t =_\alpha t'}{\mathtt{snd}\,t =_\alpha \mathtt{snd}\,t'} \qquad \frac{s =_\alpha s' \qquad t =_\alpha t'}{s\,t =_\alpha s'\,t'}$$

$$\frac{(y\;x)\cdot t =_\alpha (y\;x')\cdot t' \qquad y \text{ does not occur in } \{x, x', t, t'\}}{\lambda x : A.\,t =_\alpha \lambda x' : A.\,t'}$$

result of replacing all occurrences of x with y in t

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$$\frac{t =_\alpha t'}{\texttt{snd}\,t =_\alpha \texttt{snd}\,t'} \qquad \frac{s =_\alpha s' \qquad t =_\alpha t'}{s\,t =_\alpha s'\,t'}$$

$$\frac{(y\ x)\cdot t =_\alpha (y\ x')\cdot t' \qquad y \text{ does not occur in } \{x, x', t, t'\}}{\lambda x : A.\, t =_\alpha \lambda x' : A.\, t'}$$

E.g.

$\lambda x : A.\, x\,x =_\alpha \lambda y : A.\, y\,y \neq_\alpha \lambda x : A.\, x\,y$

$(\lambda y : A.\, y)\,x =_\alpha (\lambda x : A.\, x)\,x \neq_\alpha (\lambda x : A.\, x)\,y$

Substitution

$t[s/x]$

$=$ result of replacing all free occurrences of variable x in term t (i.e. those not occurring within the scope of a $\lambda x : A.\_$ binder) by the term s, alpha-converting λ-bound variables in t to avoid them "capturing" any free variables of t.

E.g. $(\lambda y : A.\,(y\,,x))[y/x]$ is $\lambda z : A.\,(z\,,y)$ and is not $\lambda y : A.\,(y\,,y)$

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The relation $t[s/x] = t'$ can be inductively defined by the following rules …

Substitution

$$\frac{}{c^A[s/x] = c^A} \qquad \frac{}{x[s/x] = s} \qquad \frac{y \neq x}{y[s/x] = y} \qquad \frac{}{()[s/x] = ()}$$

$$\frac{t_1[s/x] = t_1' \qquad t_2[s/x] = t_2'}{(t_1\ ,\ t_2)[s/x] = (t_1'\ ,\ t_2')} \qquad \frac{t[s/x] = t'}{(\mathtt{fst}\ t)[s/x] = \mathtt{fst}\ t'}$$

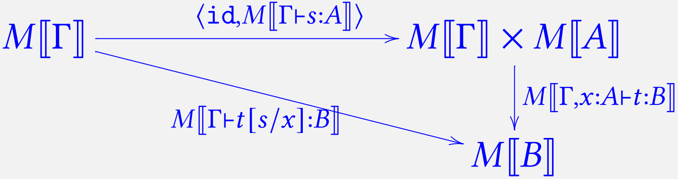
$$\frac{t[s/x] = t'}{(\mathtt{snd}\ t)[s/x] = \mathtt{snd}\ t'} \qquad \frac{t_1[s/x] = t_1' \qquad t_2[s/x] = t_2'}{(t_1\ t_2)[s/x] = t_1' t_2'}$$

$$\frac{t[s/x] = t' \qquad y \neq x \text{ and } y \text{ does not occur in } s}{(\lambda y : A.\, t)[s/x] = \lambda y : A.\, t'}$$

Semantics of substitution in a ccc

Substitution Lemma If $\Gamma \vdash s : A$ and $\Gamma, x : A \vdash t : B$ are provable, then so is $\Gamma \vdash t[s/x] : B$.

Substitution Theorem If $\Gamma \vdash s : A$ and $\Gamma, x : A \vdash t : B$ are provable, then in any ccc the following diagram commutes:



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Qu: which equations are always satisfied in any ccc?

Ans: $\beta\eta$-equivalence...

STLC $\beta\eta$-Equality

The relation $\boxed{\Gamma \vdash s =_{\beta\eta} t : A}$ (where Γ ranges over typing environments, s and t over terms, and A over types) is inductively defined by the following rules:

STLC $\beta\eta$-Equality

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- β-conversions

$$\frac{\Gamma, x : A \vdash t : B \qquad \Gamma \vdash s : A}{\Gamma \vdash (\lambda x : A.\, t)s =_{\beta\eta} t[s/x] : B}$$

$$\frac{\Gamma \vdash s : A \qquad \Gamma \vdash t : B}{\Gamma \vdash \mathtt{fst}(s\,,\,t) =_{\beta\eta} s : A}$$

$$\frac{\Gamma \vdash s : A \qquad \Gamma \vdash t : B}{\Gamma \vdash \mathtt{snd}(s\,,\,t) =_{\beta\eta} t : B}$$

STLC $\beta\eta$-Equality

The relation $\Gamma \vdash s =_{\beta\eta} t : A$ (where Γ ranges over typing environments, s and t over terms, and A over types) is inductively defined by the following rules:

- β-conversions
- η-conversions

$$\frac{\Gamma \vdash t : A \to B \qquad x \text{ does not occur in } t}{\Gamma \vdash t =_{\beta\eta} (\lambda x : A.\, t\, x) : A \to B}$$

$$\frac{\Gamma \vdash t : A \times B}{\Gamma \vdash t =_{\beta\eta} (\mathtt{fst}\ t\, ,\, \mathtt{snd}\ t) : A \times B}$$

$$\frac{\Gamma \vdash t : \mathtt{unit}}{\Gamma \vdash t =_{\beta\eta} () : \mathtt{unit}}$$

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The relation $\Gamma \vdash s =_{\beta\eta} t : A$ (where Γ ranges over typing environments, s and t over terms, and A over types) is inductively defined by the following rules:

- β-conversions
- η-conversions
- congruence rules

$$\frac{\Gamma, x : A \vdash t =_{\beta\eta} t' : B}{\Gamma \vdash \lambda x : A.\, t =_{\beta\eta} \lambda x : A.\, t' : A \to B}$$

$$\frac{\Gamma \vdash s =_{\beta\eta} s' : A \to B \qquad \Gamma \vdash t =_{\beta\eta} t' : A}{\Gamma \vdash s\, t =_{\beta\eta} s'\, t' : B}$$ etc

STLC $\beta\eta$-Equality

The relation $\Gamma \vdash s =_{\beta\eta} t : A$ (where Γ ranges over typing environments, s and t over terms, and A over types) is inductively defined by the following rules:

- β-conversions
- η-conversions
- congruence rules
- $=_{\beta\eta}$ is reflexive, symmetric and transitive

$$\frac{\Gamma \vdash t : A}{\Gamma \vdash t =_{\beta\eta} t : A} \qquad \frac{\Gamma \vdash s =_{\beta\eta} t : A}{\Gamma \vdash t =_{\beta\eta} s : A}$$

$$\frac{\Gamma \vdash r =_{\beta\eta} s : A \qquad \Gamma \vdash s =_{\beta\eta} t : A}{\Gamma \vdash r =_{\beta\eta} t : A}$$

STLC $\beta\eta$-Equality

Soundness Theorem for semantics of STLC in a ccc.
If $\Gamma \vdash s =_{\beta\eta} t : A$ is provable, then in any ccc

$$M[\![\Gamma \vdash s : A]\!] = M[\![\Gamma \vdash t : A]\!]$$

are equal $\mathbf{C}$-morphisms $M[\![\Gamma]\!] \longrightarrow M[\![A]\!]$.

Proof is by induction on the structure of the proof of $\Gamma \vdash s =_{\beta\eta} t : A$.

Here we just check the case of β-conversion for functions.

So suppose we have $\Gamma, x : A \vdash t : B$ and $\Gamma \vdash s : A$. We have to see that

$$M[\![\Gamma \vdash (\lambda x : A.\, t)\, s : B]\!] = M[\![\Gamma \vdash t[s/x] : B]\!]$$

Suppose

$$\begin{aligned} M[\![\Gamma]\!] &= X \\ M[\![A]\!] &= Y \\ M[\![B]\!] &= Z \\ M[\![\Gamma, x : A \vdash t : B]\!] &= f : X \times Y \longrightarrow Z \\ M[\![\Gamma \vdash s : A]\!] &= g : X \longrightarrow Z \end{aligned}$$

Then

$$M[\![\Gamma \vdash \lambda x : A.\, t : A \to B]\!] = \mathtt{cur}\, f : X \longrightarrow Z^Y$$

and hence

$$\begin{aligned} & M[\![\Gamma \vdash (\lambda x : A.\, t)\, s : B]\!] \\ & = \mathtt{app} \circ \langle \mathtt{cur}\, f\, , g \rangle \\ & = \mathtt{app} \circ (\mathtt{cur}\, f \times \mathtt{id}_Y) \circ \langle \mathtt{id}_X\, , g \rangle && \text{since } (a \times b) \circ \langle c\, , d \rangle = \langle a \circ c\, , b \circ d \rangle \\ & = f \circ \langle \mathtt{id}_X\, , g \rangle && \text{by definition of } \mathtt{cur}\, f \\ & = M[\![\Gamma \vdash t[s/x] : B]\!] && \text{by the Substitution Theorem} \end{aligned}$$

as required.

The internal language of a ccc, $\mathbf{C}$

- one ground type for each $\mathbf{C}$-object X
- for each $X \in \mathbf{C}$, one constant f^X for each $\mathbf{C}$-morphism $f : 1 \to X$ ("global element" of the object X)

The types and terms of STLC over this language usefully describe constructions on the objects and morphisms of $\mathbf{C}$ using its cartesian closed structure, but in an "element-theoretic" way.

For example, …

Example

In any ccc $\mathbf{C}$, for any $X, Y, Z \in \mathbf{C}$ there is an isomorphism

$$\boxed{Z^{(X \times Y)} \cong (Z^Y)^X}$$

Example

In any ccc $\mathbf{C}$, for any $X, Y, Z \in \mathbf{C}$ there is an isomorphism

$$Z^{(X \times Y)} \cong (Z^Y)^X$$

which in the internal language of $\mathbf{C}$ is described by the terms

$$\diamond \vdash s : ((X \times Y) \to Z) \to (X \to (Y \to Z))$$
$$\diamond \vdash t : (X \to (Y \to Z)) \to ((X \times Y) \to Z)$$

where $\begin{cases} s & \triangleq \lambda f : (X \times Y) \to Z.\, \lambda x : X.\, \lambda y : Y.\, f(x\,,y) \\ t & \triangleq \lambda g : X \to (Y \to Z).\, \lambda z : X \times Y.\, g\,(\texttt{fst}\ z)\,(\texttt{snd}\ z) \end{cases}$ and

which satisfy $\begin{cases} \diamond, f : (X \times Y) \to Z \vdash t(s\,f) =_{\beta\eta} f \\ \diamond, g : X \to (Y \to Z) \vdash s(t\,g) =_{\beta\eta} g \end{cases}$

Free cartesian closed categories

The Soundness Theorem has a converse—completeness.

In fact for a given set of ground types and typed constants there is a single ccc $\boxed{\mathbf{F}}$ (the free ccc for that language) with an interpretation function M so that $\Gamma \vdash s =_{\beta\eta} t : A$ is provable iff $M[\![\Gamma \vdash s : A]\!] = M[\![\Gamma \vdash t : A]\!]$ in $\mathbf{F}$.

Free cartesian closed categories

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- $\mathbf{F}$-objects are the STLC types over the given set of ground types
- $\mathbf{F}$-morphisms $A \longrightarrow B$ are equivalence classes of STLC terms t satisfying $\diamond \vdash t : A \to B$ (so t is a *closed* term—it has no free variables) with respect to the equivalence relation equating s and t if $\diamond \vdash s =_{\beta\eta} t : A \to B$ is provable.
- identity morphism on A is the equivalence class of $\diamond \vdash \lambda x : A.\, x : A \to A$.
- composition of a morphism $A \longrightarrow B$ represented by $\diamond \vdash s : A \to B$ and a morphism $B \longrightarrow C$ represented by $\diamond \vdash t : B \to C$ is represented by $\diamond \vdash \lambda x : A.\, t(s\, x) : A \to C$.

Curry-Howard correspondence

Logic		**Type Theory**
propositions	$\leftrightarrow$	types
proofs	$\leftrightarrow$	terms

E.g. IPL *versus* STLC.

Curry-Howard for IPL *vs* STLC

Proof of $\diamond, \varphi \Rightarrow \psi, \psi \Rightarrow \theta \vdash \varphi \Rightarrow \theta$ in IPL

$$
\dfrac{
  \dfrac{\dfrac{}{\cdots}\,(\text{AX})}{\Phi \vdash \psi \Rightarrow \theta}\,(\text{WK})
  \qquad
  \dfrac{
    \dfrac{\dfrac{\dfrac{}{\cdots}\,(\text{AX})}{\cdots}\,(\text{WK})}{\Phi \vdash \varphi \Rightarrow \psi}\,(\text{WK})
    \qquad
    \dfrac{}{\Phi \vdash \varphi}\,(\text{AX})
  }{\Phi \vdash \psi}\,(\Rightarrow\text{E})
}{
  \dfrac{\Phi \vdash \theta}{\diamond, \varphi \Rightarrow \psi, \psi \Rightarrow \theta \vdash \varphi \Rightarrow \theta}\,(\Rightarrow\text{I})
}\,(\Rightarrow\text{E})
$$

where $\Phi = \diamond, \varphi \Rightarrow \psi, \psi \Rightarrow \theta, \varphi$

Curry-Howard for IPL *vs* STLC

and a corresponding STLC term

$$
\dfrac{
\dfrac{\dfrac{}{\cdots}\,(\textsc{ax})}{\Phi \vdash z : \psi \Rightarrow \theta}\,(\textsc{wk})
\qquad
\dfrac{
\dfrac{\dfrac{\dfrac{}{\cdots}\,(\textsc{ax})}{\cdots}\,(\textsc{wk})}{\Phi \vdash y : \varphi \Rightarrow \psi}\,(\textsc{wk})
\qquad
\dfrac{}{\Phi \vdash x : \varphi}\,(\textsc{ax})
}{\Phi \vdash y\,x : \psi}\,(\Rightarrow\textsc{e})
}{
\dfrac{\Phi \vdash z(y\,x) : \theta}{\diamond, y : \varphi \Rightarrow \psi, z : \psi \Rightarrow \theta \vdash \lambda x : \varphi.\, z(y\,x) : \varphi \Rightarrow \theta}\,(\Rightarrow\textsc{i})
}\,(\Rightarrow\textsc{e})
$$

where $\Phi = \diamond, y : \varphi \Rightarrow \psi, z : \psi \Rightarrow \theta, x : \varphi$

Curry-Howard-Lawvere/Lambek correspondence

Logic		**Type Theory**		**Category Theory**
propositions	$\leftrightarrow$	types	$\leftrightarrow$	objects
proofs	$\leftrightarrow$	terms	$\leftrightarrow$	morphisms

E.g. IPL *versus* STLC *versus* CCCs

Curry-Howard-Lawvere/Lambek correspondence

Logic		Type Theory		Category Theory
propositions	$\leftrightarrow$	types	$\leftrightarrow$	objects
proofs	$\leftrightarrow$	terms	$\leftrightarrow$	morphisms

E.g. IPL *versus* STLC *versus* CCCs

These correspondences can be made into category-theoretic equivalences—we first need to define the notions of functor and natural transformation in order to define the notion of equivalence of categories.

Lecture 10

Functors

are the appropriate notion of morphism between categories

Given categories $\mathbf{C}$ and $\mathbf{D}$, a functor $F : \mathbf{C} \to \mathbf{D}$ is specified by:

- a function $\mathtt{obj}\,\mathbf{C} \to \mathtt{obj}\,\mathbf{D}$ whose value at X is written $F\,X$
- for all $X, Y \in \mathbf{C}$, a function $\mathbf{C}(X, Y) \to \mathbf{D}(F\,X, F\,Y)$ whose value at $f : X \to Y$ is written $F\,f : F\,X \to F\,Y$ and which is required to preserve composition and identity morphisms:

$$
\begin{aligned}
F(g \circ f) &= F\,g \circ F\,f \\
F(\mathtt{id}_X) &= \mathtt{id}_{F\,X}
\end{aligned}
$$