University of Cambridge
2024/25 Part II / Part III / MPhil ACS
Category Theory
Exercise Sheet 1
by Andrew Pitts and Marcelo Fiore

Sets

1. For a set I and an I-indexed family of sets $\{ X_i \}_{i \in I}$, define their
   (a) product $\prod_{i \in I} X_i$ with projection functions $\{ \pi_k : \prod_{i \in I} X_i \to X_k \}_{k \in I}$, and
   (b) sum $\sum_{i \in I} X_i$ with tagging functions $\{ \iota_k : X_k \to \sum_{i \in I} X_i \}_{k \in I}$.

2. (a) Show that for all functions between sets $X \xleftarrow{f} Z \xrightarrow{g} Y$, there exists a unique function $\langle f, g \rangle : Z \to X \times Y$ such that $\pi_1 \circ \langle f, g \rangle = f$ and $\pi_2 \circ \langle f, g \rangle = g$.
   Generalise this statement from binary to I-indexed products.
   (b) For functions $f : A \to X$ and $g : B \to Y$, give an explicit description of the function $f \times g \triangleq \langle f \circ \pi_1 \, , \, g \circ \pi_2 \rangle : A \times B \to X \times Y$.
   Show that $\mathtt{id}_A \times \mathtt{id}_B = \mathtt{id}_{A \times B}$ and that, for $p : X \to U$ and $q : Y \to V$, $(p \times q) \circ (f \times g) = (p \circ f) \times (q \circ g) : A \times B \to U \times V$.

3. (a) Show that for all functions between sets $X \xrightarrow{f} Z \xleftarrow{g} Y$, there exists a unique function $[f, g] : X + Y \to Z$ such that $[f, g] \circ \iota_1 = f$ and $[f, g] \circ \iota_2 = g$.
   Generalise this statement from binary to I-indexed sums.
   (b) For functions $f : A \to X$ and $g : B \to Y$, give and explicit description of the function $f + g \triangleq [\iota_1 \circ f \, , \, \iota_2 \circ g] : A + B \to X + Y$.
   Show that $\mathtt{id}_A + \mathtt{id}_B = \mathtt{id}_{A+B}$ and that, for $p : X \to U$ and $q : Y \to V$, $(p + q) \circ (f + g) = (p \circ f) + (q \circ g) : A + B \to U + V$.

4. (a) Show that the sets $2 = \{0, 1\}$ and $3 = \{0, 1, 2\}$ are not isomorphic; that is, there is no isomorphism between them.
   (b) Why are the sets $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ (integers) and $\mathbb{Q}$ (rational numbers) isomorphic?

5. Exhibit as many as possible isomorphisms as you can find between expressions built up from arbitrary sets X, Y, Z, the sets $1, 0$, and the constructions $\times, \Rightarrow, +$. For instance, $(X \times Y) \times Z \cong X \times (Y \times Z)$.

6. A function $f : X \to Y$ is *injective* whenever for all $x, x' \in X$, $f(x) = f(x')$ implies $x = x'$.
   A function $f : X \to Y$ is a *monomorphism* whenever for every set Z and every pair of morphisms $g, h : Z \to X$ we have
   $$f \circ g = f \circ h \implies g = h$$
   Show that a function is injective if, and only if, it is a monomorphism.

7. A function $f : X \longrightarrow Y$ is *surjective* whenever for all $y \in Y$ there exists $x \in X$ such that $f(x) = y$.

   A function $f : X \longrightarrow Y$ is an *epimorrphism* whenever for every set Z and every pair of morphisms $g, h : Y \longrightarrow Z$ we have

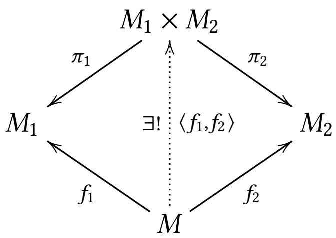
   $$g \circ f = h \circ f \;\Rightarrow\; g = h$$

   Show that a function is surjective if, and only if, it is an epimorphism.

Monoids

1. Show that for all monoid (resp. group) homomorphisms between monoids (resp. groups) $M_1 \xleftarrow{f_1} M \xrightarrow{f_2} M_2$, there exists a unique monoid (resp. group) homomorphism $\langle f_1, f_2 \rangle : M \longrightarrow M_1 \times M_2$ such that $\pi_1 \circ \langle f_1, f_2 \rangle = f_1$ and $\pi_2 \circ \langle f_1, f_2 \rangle = f_2$.

   In diagrammatic form:



2. Consider the following monoids and homorphisms between them

   $$\texttt{List}(X_1) \xrightarrow{\texttt{map}\,\iota_1} \texttt{List}(X_1 + X_2) \xleftarrow{\texttt{map}\,\iota_2} \texttt{List}(X_2)$$

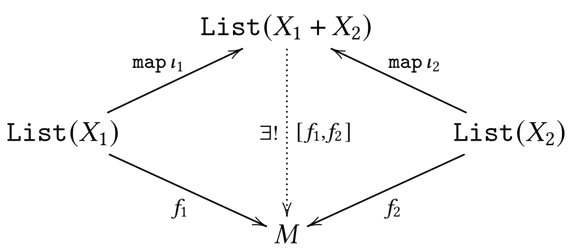
   Show that for all monoids M and monoid homomorphisms as follows

   $$\texttt{List}(X_1) \xrightarrow{f_1} M \xleftarrow{f_2} \texttt{List}(X_2)$$

   there exists a unique monoid homomorphism $[f_1, f_2] : \texttt{List}(X_1 + X_2) \longrightarrow M$ such that

   $$[f_1, g_1] \circ \texttt{map}\,\iota_1 = f_1 \text{ and } [f_1, f_2] \circ \texttt{map}\,\iota_2 = f_2$$

   In diagrammatic form:



Groups

1. (a) Show that if $(G, \overline{}, \bullet, \iota)$ and $(G, \overline{}, \bullet, \iota')$ are groups, then $\iota = \iota'$.

   (b) Show that if $(G, \overline{}, \bullet, \iota)$ and $(G, \overline{}', \bullet, \iota)$ are groups, then $\overline{} = \overline{}'$.

2. An endofunction $f : X \longrightarrow X$ is an *involution* whenever $f \circ f = \mathtt{id}_X$.

   For a group $(G, \overline{}, \bullet, \iota)$, show that $\overline{}$ is an involution.

3. For a group $(G, \overline{}, \bullet, \iota)$, show that:

   (a) for all $x, y \in G$, $x \bullet y = \iota$ implies $y = \overline{x}$ and $x = \overline{y}$;

   (b) $\overline{\iota} = \iota$;

   (c) for all $x, y \in G$, $\overline{(x \bullet y)} = \overline{y} \bullet \overline{x}$.

Universal problems

1. Let X be a set and consider a monoid $\underline{F}X = (FX, \bullet_X, \iota_X)$ together with a function $\varphi_X : X \longrightarrow FX$.

   Observe that $\underline{F}X$ and φ_X are a solution to the problem of freely generating a monoid from the set X if, and only if, for all monoids $\underline{M} = (M, \bullet, \iota)$ the function

   $$\_ \circ \varphi_X : \mathbf{Mon}(\underline{F}X, \underline{M}) \longrightarrow \mathbf{Set}(X, M) : h \mapsto h \circ \varphi_X$$

   is bijective.

   Derive the following *proof technique*:

   > For all monoids $\underline{M}$ and monoid homomorphisms $f, g : \underline{F}X \longrightarrow \underline{M}$,
   >
   > $$f = g : FX \longrightarrow M \text{ if, and only if, } f \circ \varphi_X = g \circ \varphi_X : X \longrightarrow M.$$

2. (a) For a set A, let $\mathtt{s}_A : A \longrightarrow \mathtt{List}\,A$ be the function given, for all $a \in A$, by

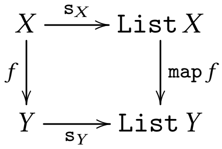
   $$\mathtt{s}_A(a) = [a] \triangleq (a :: \mathtt{nil})$$

   Show that $(\mathtt{List}\,A, @_A, \mathtt{nil}_A)$ and $\mathtt{s}_A : A \longrightarrow \mathtt{List}\,A$ are a solution to the problem of freely generating a monoid from the set A.

   (b) For a function $f : X \longrightarrow Y$, define the monoid homomorphism $\mathtt{map}\,f : \mathtt{List}(X) \longrightarrow \mathtt{List}(Y)$ as

   $$\mathtt{map}\,f \triangleq (\mathtt{s}_Y \circ f)^{\#}$$

   Observe that, by definition, the diagram below commutes:



   Show that:

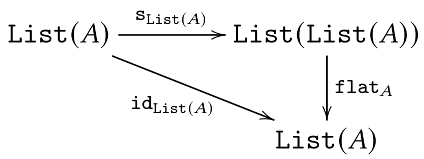
   i. $\mathtt{map}\,\mathtt{id}_X = \mathtt{id}_{\mathtt{List}(X)}$, and

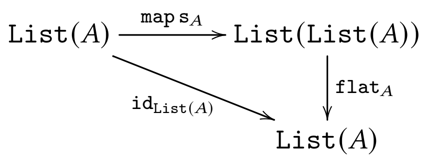
ii. for all functions $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$, $\mathtt{map}(g \circ f) = \mathtt{map}\, g \circ \mathtt{map}\, f$.

(c) Let

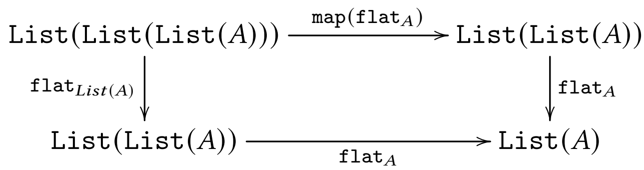
$$\mathtt{flat}_A \triangleq (\mathtt{id}_{\mathtt{List}(A)})^{\#} : \mathtt{List}(\mathtt{List}\, A) \longrightarrow \mathtt{List}(A)$$

Observe that, by definition, the diagram on the left below commutes





Show that the diagram on the right above and the diagram below also commute.



[Hint: Use the above proof technique.]

3. Freely generating a monoid from a pointed set.

A *pointed set* is a structure $\underline{X} = (X, x)$ consisting of a set X and an element $x \in X$. A pointed-set homomorphism $h : (X, x) \longrightarrow (Y, y)$ between pointed sets is a function $h : X \longrightarrow Y$ such that $h(x) = y$.

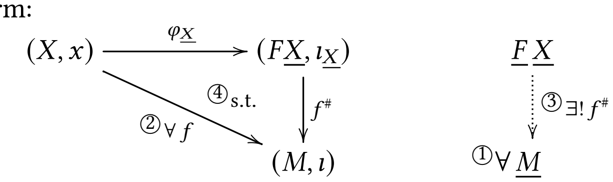
Given a pointed set $\underline{X} = (X, x)$,

(a) construct a monoid $\underline{F}\,\underline{X} = (F\underline{X}, \bullet_{\underline{X}}, \iota_{\underline{X}})$ and a pointed-set homomorphism $\varphi_{\underline{X}} : (X, x) \longrightarrow (F\underline{X}, \iota_{\underline{X}})$

such that

(a) for all monoids $\underline{M} = (M, \bullet, \iota)$ and all pointed-set homomorphisms $f : (X, x) \longrightarrow (M, \iota)$, there exists a unique monoid homomorphism $f^{\#} : \underline{F}\,\underline{X} \longrightarrow \underline{M}$ such that $f^{\#} \circ \varphi_{\underline{X}} = f$.

In diagrammatic form:



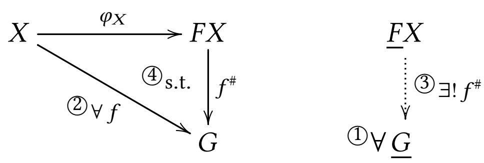
4. Given a set X,

(a) construct a group $\underline{F}X = (FX, ^{-X}, \bullet_X, \iota_X)$ and a function $\varphi_X : X \longrightarrow FX$

such that

(a) for all groups $\underline{G} = (G, ^{-}, \bullet, \iota)$ and all functions $f : X \longrightarrow G$, there exists a unique group homomorphism $f^{\#} : \underline{F}X \longrightarrow \underline{G}$ such that $f^{\#} \circ \varphi_X = f$.

In diagrammatic form:



Categories

1. For $f : X \to Y$ and $g, h : Y \to X$, show that if $g \circ f = \mathtt{id}_X$ and $f \circ h = \mathtt{id}_Y$ then $g = h$.

2. Let **C** be a category. A morphism $f : X \to Y$ in **C** is called a *monomorphism*, if for every object $Z \in \mathbf{C}$ and every pair of morphisms $g, h : Z \to X$ we have

$$f \circ g = f \circ h \implies g = h$$

   It is called a *split monomorphism* if there is some morphism $g : Y \to X$ with $g \circ f = \mathtt{id}_X$, in which case we say that g is a *left inverse* for f.

   (a) Prove that every split monomorphism is a monomorphism.
   (b) Prove that if $f : X \to Y$ and $g : Y \to Z$ are monomorphisms then $g \circ f : X \to Z$ is a monomorphism.
   (c) Prove that, for morphisms $f : X \to Y$ and $g : Y \to Z$, if $g \circ f$ is a monomorphism then f is a monomorphism.
   (d) Is every monomorphism in **Set** a split monomorphism?
   (e) Show that a split monomorphism can have more than one left inverse.

3. Let **C** be a category. A morphism $f : X \to Y$ in **C** is called an *epimorphism*, if for every object $Z \in \mathbf{C}$ and every pair of morphisms $g, h : Y \to Z$ we have

$$g \circ f = h \circ f \implies g = h$$

   It is called a *split epimorphism* if there is some morphism $g : Y \to X$ with $f \circ g = \mathtt{id}_Y$, in which case we say that g is a *right inverse* for f.

   (a) Prove that every split epimorphism is an epimorphism.
   (b) Prove that if $f : X \to Y$ and $g : Y \to Z$ are epimorphisms then $g \circ f : X \to Z$ is an epimorphism.
   (c) Prove that, for morphisms $f : X \to Y$ and $g : Y \to Z$, if $g \circ f$ is a epimorphism then g is an epimorphism.
   (d) Is every epimorphism in **Set** a split epimorphism?
   (e) Show that a split epimorphism can have more than one right inverse.

Isomorphism

1. Let **Mat** be a category whose objects are the positive natural numbers and whose morphisms $M \in \mathbf{Mat}(m, n)$ are $m \times n$ matrices with real number entries. If composition is given by matrix multiplication, what are the identity morphisms? Give an example of an isomorphism in **Mat** that is not an identity. Can two objects m and n be isomorphic in **Mat** if $m \neq n$?

2. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms in a category.

   (a) Prove that if f and g are both isomorphisms, with inverses f^{-1} and g^{-1} respectively, then $g \circ f$ is an isomorphism and its inverse is $f^{-1} \circ g^{-1}$.

   (b) Prove that if f and $g \circ f$ are both isomorphisms then so is g.

   (c) If $g \circ f$ is an isomorphism, does that necessarily imply that either of f or g are isomorphisms?

3. Give an example of a category containing a morphism that is both a monomorphism and an epimorphism, but not an isomorphism.