Randomised Algorithms

Lecture 6: Linear Programming: Introduction

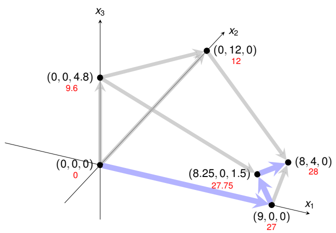
Thomas Sauerwald (`tms41@cam.ac.uk`)

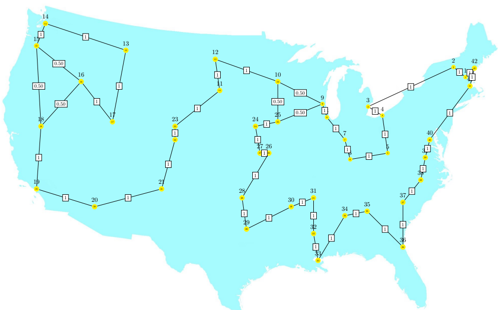
Lent 2025

UNIVERSITY OF CAMBRIDGE

Outline

Introduction





- linear programming is a powerful tool in optimisation
- inspired more sophisticated techniques such as quadratic optimisation, convex optimisation, integer programming and semi-definite programming
- we will later use the connection between linear and integer programming

Overall we will approach the following problems with linear programming:

1. a "generic" production problem, shortest path, maximum flow, minimum-cost flow (directly)
2. TSP, Vertex Cover, Set Cover, MAX-CNF (indirectly)

Outline

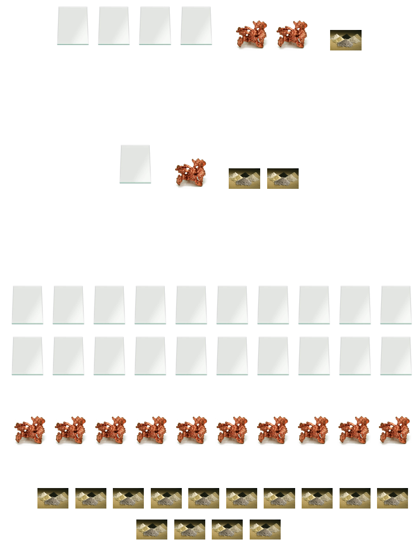
What are Linear Programs?

Linear Programming (informal definition)

- maximise or minimise an objective, given limited resources (competing constraints)
- constraints are specified as (in)equalities
- objective function and constraints are linear

A Simple Example of a Linear Optimisation Problem

- Laptop
  - selling price to retailer: 1,000 GBP
  - glass: 4 units
  - copper: 2 units
  - rare-earth elements: 1 unit
- Smartphone
  - selling price to retailer: 1,000 GBP
  - glass: 1 unit
  - copper: 1 unit
  - rare-earth elements: 2 units
- You have a daily supply of:
  - glass: 20 units
  - copper: 10 units
  - rare-earth elements: 14 units
  - (and enough of everything else...)



How to maximise your daily earnings?

The Linear Program

Linear Program for the Production Problem

$$\begin{array}{llcrcl} \text{maximise} & x_1 & + & x_2 & & \\ \text{subject to} & & & & & \\ & 4x_1 & + & x_2 & \leq & 20 \\ & 2x_1 & + & x_2 & \leq & 10 \\ & x_1 & + & 2x_2 & \leq & 14 \\ & x_1, x_2 & & & \geq & 0 \end{array}$$

The solution of this linear program yields the optimal production schedule.

Formal Definition of Linear Program

- Given $a_1, a_2, \ldots, a_n$ and a set of variables $x_1, x_2, \ldots, x_n$, a linear function f is defined by

$$f(x_1, x_2, \ldots, x_n) = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n.$$

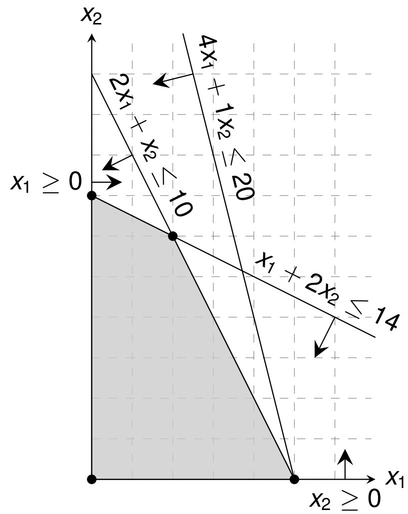
- Linear Equality: $f(x_1, x_2, \ldots, x_n) = b$
- Linear Inequality: $f(x_1, x_2, \ldots, x_n) \gtreqless b$
- Linear-Progamming Problem: either minimise or maximise a linear function subject to a set of linear constraints

Linear Constraints

Finding the Optimal Production Schedule

$$\begin{array}{llcrcl} \text{maximise} & x_1 & + & x_2 & & \\ \text{subject to} & & & & & \\ & 4x_1 & + & x_2 & \leq & 20 \\ & 2x_1 & + & x_2 & \leq & 10 \\ & x_1 & + & 2x_2 & \leq & 14 \\ & x_1, x_2 & & & \geq & 0 \end{array}$$

Any setting of x_1 and x_2 satisfying all constraints is a feasible solution

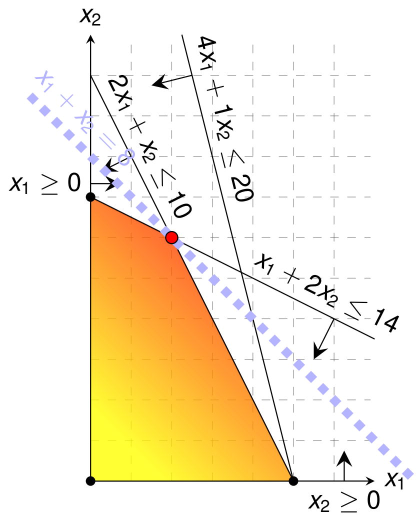


Question: Which aspect did we ignore in the formulation of the linear program?

Finding the Optimal Production Schedule

maximise $x_1 + x_2$
subject to

$$\begin{array}{rcrcl} 4x_1 & + & x_2 & \leq & 20 \\ 2x_1 & + & x_2 & \leq & 10 \\ x_1 & + & 2x_2 & \leq & 14 \\ x_1, x_2 & & & \geq & 0 \end{array}$$



Graphical Procedure: Move the line $x_1 + x_2 = z$ as far up as possible.

While the same approach also works for higher-dimensions, we need to take a more systematic and algebraic procedure.

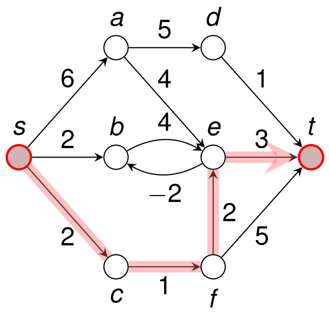
Outline

Shortest Paths

Single-Pair Shortest Path Problem

- Given: directed graph $G = (V, E)$ with edge weights $w : E \to \mathbb{R}$, pair of vertices $s, t \in V$
- Goal: Find a path of minimum weight from s to t in G

$p = (v_0 = s, v_1, \ldots, v_k = t)$ such that $w(p) = \sum_{i=1}^{k} w(v_{k-1}, v_k)$ is minimised.



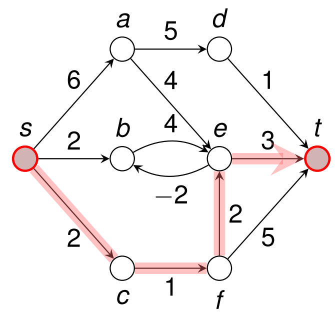
Exercise: Translate the SPSP problem into a linear program which finds the distance between s and v!

Shortest Paths

Single-Pair Shortest Path Problem

- Given: directed graph $G = (V, E)$ with edge weights $w : E \to \mathbb{R}$, pair of vertices $s, t \in V$
- Goal: Find a path of minimum weight from s to t in G

$p = (v_0 = s, v_1, \ldots, v_k = t)$ such that $w(p) = \sum_{i=1}^{k} w(v_{k-1}, v_k)$ is minimised.



Shortest Paths as LP

maximise d_t
subject to

$$\begin{array}{rcll} d_v & \leq & d_u + w(u,v) & \text{for each edge } (u,v) \in E, \\ d_s & = & 0. & \end{array}$$

Recall: When BELLMAN-FORD terminates, all these inequalities are satisfied.

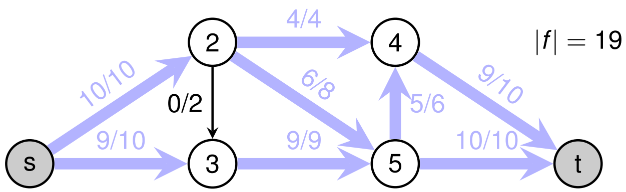
this is a maximisation problem!

Solution $\overline{d}$ satisfies $\overline{d}_v = \min_{u:\,(u,v)\in E}\{\overline{d}_u + w(u,v)\}$

Maximum Flow

Maximum Flow Problem

- Given: directed graph $G = (V, E)$ with edge capacities $c : E \to \mathbb{R}^+$ (recall $c(u, v) = 0$ if $(u, v) \notin E$), pair of vertices $s, t \in V$
- Goal: Find a maximum flow $f : V \times V \to \mathbb{R}$ from s to t which satisfies the capacity constraints and flow conservation



Maximum Flow as LP

$$
\begin{array}{lrcll}
\text{maximise} & \sum_{v\in V} f_{sv} & - & \sum_{v\in V} f_{vs} & \\
\text{subject to} & & & & \\
& f_{uv} & \leq & c(u,v) & \text{for each } u, v \in V, \\
& \sum_{v\in V} f_{vu} & = & \sum_{v\in V} f_{uv} & \text{for each } u \in V \setminus \{s,t\}, \\
& f_{uv} & \geq & 0 & \text{for each } u, v \in V.
\end{array}
$$

Minimum-Cost Flow

Extension of the Maximum Flow Problem

Minimum-Cost-Flow Problem

- Given: directed graph $G = (V, E)$ with capacities $c : E \to \mathbb{R}^+$, pair of vertices $s, t \in V$, cost function $a : E \to \mathbb{R}^+$, flow demand of d units
- Goal: Find a flow $f : V \times V \to \mathbb{R}$ from s to t with $|f| = d$ while minimising the total cost $\sum_{(u,v)\in E} a(u, v) f_{uv}$ incurrred by the flow.

Optimal Solution with total cost:
$\sum_{(u,v)\in E} a(u, v) f_{uv} = (2{\cdot}2)+(5{\cdot}2)+(3{\cdot}1)+(7{\cdot}1)+(1{\cdot}3) = 27$

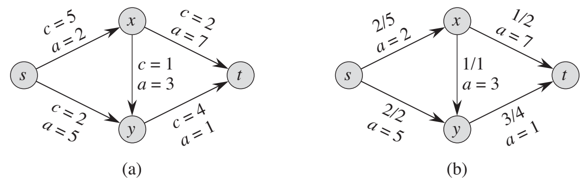


Figure 29.3 **(a)** An example of a minimum-cost-flow problem. We denote the capacities by c and the costs by a. Vertex s is the source and vertex t is the sink, and we wish to send 4 units of flow from s to t. **(b)** A solution to the minimum-cost flow problem in which 4 units of flow are sent from s to t. For each edge, the flow and capacity are written as flow/capacity.

Minimum-Cost Flow as a LP

Minimum-Cost Flow as LP

$$
\begin{array}{lrcll}
\text{minimise} & \sum_{(u,v)\in E} a(u,v) f_{uv} & & & \\
\text{subject to} & & & & \\
& f_{uv} & \leq & c(u,v) & \text{for } u, v \in V, \\
& \sum_{v\in V} f_{vu} - \sum_{v\in V} f_{uv} & = & 0 & \text{for } u \in V \setminus \{s,t\}, \\
& \sum_{v\in V} f_{sv} - \sum_{v\in V} f_{vs} & = & d\,, & \\
& f_{uv} & \geq & 0 & \text{for } u, v \in V.
\end{array}
$$

Real power of Linear Programming comes from the ability to solve **new problems**!

???

Question: Can we use a similar approach to solve the shortest path problem?

Outline

Standard and Slack Forms

Standard Form

$$\begin{aligned} \text{maximise} \quad & \sum_{j=1}^{n} c_j x_j \\ \text{subject to} \quad & \\ & \sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad \text{for } i = 1, 2, \ldots, m \\ & x_j \geq 0 \quad \text{for } j = 1, 2, \ldots, n \end{aligned}$$

Objective Function

$n + m$ constraints

Non-Negativity Constraints

Standard Form (Matrix-Vector-Notation)

$$\begin{aligned} \text{maximise} \quad & c^T x \\ \text{subject to} \quad & \\ & Ax \leq b \\ & x \geq 0 \end{aligned}$$

Inner product of two vectors

Matrix-vector product

Converting Linear Programs into Standard Form

Reasons for a LP not being in standard form:

1. The objective might be a minimisation rather than maximisation.
2. There might be variables without nonnegativity constraints.
3. There might be equality constraints.
4. There might be inequality constraints (with $\geq$ instead of $\leq$).

Goal: Convert linear program into an equivalent program which is in standard form

Equivalence: a correspondence (not necessarily a bijection) between solutions.

Converting into Standard Form (1/5)

Reasons for a LP not being in standard form:

1. The objective might be a minimisation rather than maximisation.

$$\begin{aligned} \text{minimise} \quad & -2x_1 + 3x_2 \\ \text{subject to} \quad & \\ & x_1 + x_2 = 7 \\ & x_1 - 2x_2 \leq 4 \\ & x_1 \geq 0 \end{aligned}$$

Negate objective function

$$\begin{aligned} \text{maximise} \quad & 2x_1 - 3x_2 \\ \text{subject to} \quad & \\ & x_1 + x_2 = 7 \\ & x_1 - 2x_2 \leq 4 \\ & x_1 \geq 0 \end{aligned}$$

Converting into Standard Form (2/5)

Reasons for a LP not being in standard form:

2. There might be variables without nonnegativity constraints.

$$\begin{aligned} \text{maximise} \quad & 2x_1 - 3x_2 \\ \text{subject to} \quad & \\ & x_1 + x_2 = 7 \\ & x_1 - 2x_2 \leq 4 \\ & x_1 \geq 0 \end{aligned}$$

Replace x_2 by the difference of two non-negative variables x_2' and x_2''

$$\begin{aligned} \text{maximise} \quad & 2x_1 - 3x_2' + 3x_2'' \\ \text{subject to} \quad & \\ & x_1 + x_2' - x_2'' = 7 \\ & x_1 - 2x_2' + 2x_2'' \leq 4 \\ & x_1, x_2', x_2'' \geq 0 \end{aligned}$$

Converting into Standard Form (3/5)

Reasons for a LP not being in standard form:

3. There might be equality constraints.

$$
\begin{array}{llcrcrcrcr}
\text{maximise} & 2x_1 & - & 3x_2' & + & 3x_2'' & & \\
\text{subject to} & & & & & & & \\
& x_1 & + & x_2' & - & x_2'' & = & 7 \\
& x_1 & - & 2x_2' & + & 2x_2'' & \leq & 4 \\
& \multicolumn{5}{l}{x_1, x_2', x_2''} & \geq & 0
\end{array}
$$

Replace each equality by two inequalities.

$$
\begin{array}{llcrcrcrcr}
\text{maximise} & 2x_1 & - & 3x_2' & + & 3x_2'' & & \\
\text{subject to} & & & & & & & \\
& x_1 & + & x_2' & - & x_2'' & \leq & 7 \\
& x_1 & + & x_2' & - & x_2'' & \geq & 7 \\
& x_1 & - & 2x_2' & + & 2x_2'' & \leq & 4 \\
& \multicolumn{5}{l}{x_1, x_2', x_2''} & \geq & 0
\end{array}
$$

Converting into Standard Form (4/5)

Reasons for a LP not being in standard form:

4. There might be inequality constraints (with $\geq$ instead of $\leq$).

$$
\begin{array}{llcrcrcrcr}
\text{maximise} & 2x_1 & - & 3x_2' & + & 3x_2'' & & \\
\text{subject to} & & & & & & & \\
& x_1 & + & x_2' & - & x_2'' & \leq & 7 \\
& x_1 & + & x_2' & - & x_2'' & \geq & 7 \\
& x_1 & - & 2x_2' & + & 2x_2'' & \leq & 4 \\
& \multicolumn{5}{l}{x_1, x_2', x_2''} & \geq & 0
\end{array}
$$

Negate respective inequalities.

$$
\begin{array}{llcrcrcrcr}
\text{maximise} & 2x_1 & - & 3x_2' & + & 3x_2'' & & \\
\text{subject to} & & & & & & & \\
& x_1 & + & x_2' & - & x_2'' & \leq & 7 \\
& -x_1 & - & x_2' & + & x_2'' & \leq & -7 \\
& x_1 & - & 2x_2' & + & 2x_2'' & \leq & 4 \\
& \multicolumn{5}{l}{x_1, x_2', x_2''} & \geq & 0
\end{array}
$$

Converting into Standard Form (5/5)

Rename variable names (for consistency).

$$
\begin{array}{llcrcrcrcr}
\text{maximise} & 2x_1 & - & 3x_2 & + & 3x_3 & & \\
\text{subject to} & & & & & & & \\
& x_1 & + & x_2 & - & x_3 & \leq & 7 \\
& -x_1 & - & x_2 & + & x_3 & \leq & -7 \\
& x_1 & - & 2x_2 & + & 2x_3 & \leq & 4 \\
& \multicolumn{5}{l}{x_1, x_2, x_3} & \geq & 0
\end{array}
$$

It is always possible to convert a linear program into standard form.

Converting Standard Form into Slack Form (1/3)

Goal: Convert standard form into slack form, where all constraints except for the non-negativity constraints are equalities.

For the simplex algorithm, it is more convenient to work with equality constraints.

Introducing Slack Variables

- Let $\sum_{j=1}^{n} a_{ij} x_j \leq b_i$ be an inequality constraint
- Introduce a slack variable s by

$$
\begin{aligned}
s &= b_i - \sum_{j=1}^{n} a_{ij} x_j \\
s &\geq 0.
\end{aligned}
$$

s measures the slack between the two sides of the inequality.

- Denote slack variable of the i-th inequality by x_{n+i}

Converting Standard Form into Slack Form (2/3)

$$\begin{array}{ll} \text{maximise} & 2x_1 - 3x_2 + 3x_3 \\ \text{subject to} & \\ & x_1 + x_2 - x_3 \leq 7 \\ & -x_1 - x_2 + x_3 \leq -7 \\ & x_1 - 2x_2 + 2x_3 \leq 4 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

Introduce slack variables

$$\begin{array}{ll} \text{maximise} & 2x_1 - 3x_2 + 3x_3 \\ \text{subject to} & \\ & x_4 = 7 - x_1 - x_2 + x_3 \\ & x_5 = -7 + x_1 + x_2 - x_3 \\ & x_6 = 4 - x_1 + 2x_2 - 2x_3 \\ & x_1, x_2, x_3, x_4, x_5, x_6 \geq 0 \end{array}$$

Converting Standard Form into Slack Form (3/3)

$$\begin{array}{ll} \text{maximise} & 2x_1 - 3x_2 + 3x_3 \\ \text{subject to} & \\ & x_4 = 7 - x_1 - x_2 + x_3 \\ & x_5 = -7 + x_1 + x_2 - x_3 \\ & x_6 = 4 - x_1 + 2x_2 - 2x_3 \\ & x_1, x_2, x_3, x_4, x_5, x_6 \geq 0 \end{array}$$

Use variable z to denote objective function and omit the nonnegativity constraints.

$$\begin{array}{l} z = 2x_1 - 3x_2 + 3x_3 \\ x_4 = 7 - x_1 - x_2 + x_3 \\ x_5 = -7 + x_1 + x_2 - x_3 \\ x_6 = 4 - x_1 + 2x_2 - 2x_3 \end{array}$$

This is called slack form.

Basic and Non-Basic Variables

$$\begin{array}{l} z = 2x_1 - 3x_2 + 3x_3 \\ x_4 = 7 - x_1 - x_2 + x_3 \\ x_5 = -7 + x_1 + x_2 - x_3 \\ x_6 = 4 - x_1 + 2x_2 - 2x_3 \end{array}$$

Basic Variables: $B = \{4, 5, 6\}$

Non-Basic Variables: $N = \{1, 2, 3\}$

Slack Form (Formal Definition)

Slack form is given by a tuple (N, B, A, b, c, v) so that

$$z = v + \sum_{j \in N} c_j x_j$$

$$x_i = b_i - \sum_{j \in N} a_{ij} x_j \qquad \text{for } i \in B,$$

and all variables are non-negative.

Variables/Coefficients on the right hand side are indexed by B and N.

Slack Form (Example)

$$\begin{array}{l} z = 28 - \frac{x_3}{6} - \frac{x_5}{6} - \frac{2x_6}{3} \\ x_1 = 8 + \frac{x_3}{6} + \frac{x_5}{6} - \frac{x_6}{3} \\ x_2 = 4 - \frac{8x_3}{3} - \frac{2x_5}{3} + \frac{x_6}{3} \\ x_4 = 18 - \frac{x_3}{2} + \frac{x_5}{2} \end{array}$$

Next lecture: each slack form corresponds to a "basic" solution: $x_3 = x_5 = x_6 = 0$ and so $x_1 = 8$, $x_2 = 4$ and $x_4 = 18$, with objective value 28.

Slack Form Notation

- $B = \{1, 2, 4\}$, $N = \{3, 5, 6\}$
- $$A = \begin{pmatrix} a_{13} & a_{15} & a_{16} \\ a_{23} & a_{25} & a_{26} \\ a_{43} & a_{45} & a_{46} \end{pmatrix} = \begin{pmatrix} -1/6 & -1/6 & 1/3 \\ 8/3 & 2/3 & -1/3 \\ 1/2 & -1/2 & 0 \end{pmatrix}$$
- $$b = \begin{pmatrix} b_1 \\ b_2 \\ b_4 \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \\ 18 \end{pmatrix}, \quad c = \begin{pmatrix} c_3 \\ c_5 \\ c_6 \end{pmatrix} = \begin{pmatrix} -1/6 \\ -1/6 \\ -2/3 \end{pmatrix}$$
- $v = 28$