# Randomised Algorithms

Lecture 3: Concentration Inequalities, Application to Quick-Sort, Extensions

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UNIVERSITY OF CAMBRIDGE

# Outline

Application 2: Randomised QuickSort

Extensions of Chernoff Bounds

Applications of Method of Bounded Differences

# QuickSort

QuickSort (Input $A[1], A[2], \ldots, A[n]$)

Pick an element from the array, the so-called pivot<br>
**If** $n = 0$ or $n = 1$ **then**<br>
    **return** $A$<br>
**else**<br>
    Create two subarrays $A_1$ and $A_2$ (without the pivot) such that:<br>
        $A_1$ contains the elements that are smaller than the pivot<br>
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We will now give a proof of this “well-known” result!

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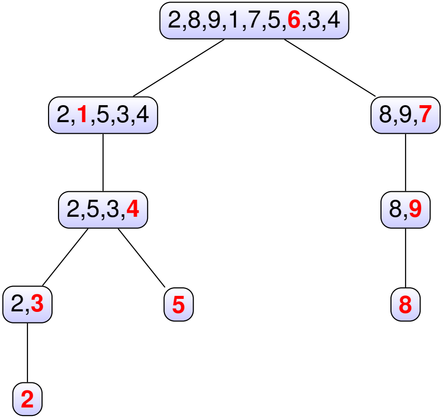

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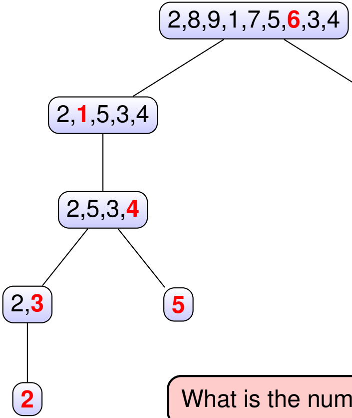


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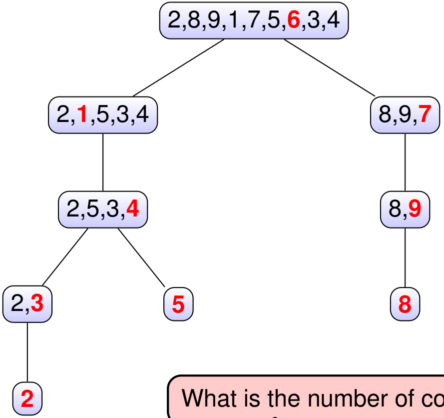


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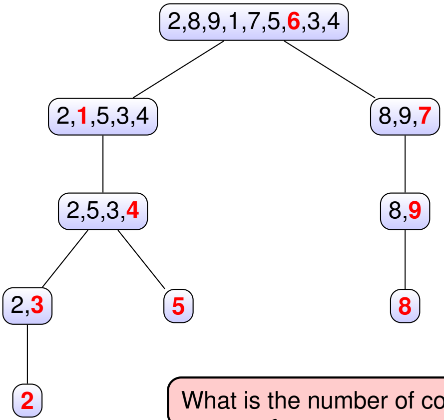


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Note that the number of comparison by QUICKSORT is equivalent to the sum of the depths of all nodes in the tree (why?). In this case:

$$0 + 1 + 1 + 2 + 2 + 3 + 3 + 3 + 4 = 19.$$

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$$\mathbf{P}[H \leq Cn \log n] \geq 1 - n^{-1}.$$

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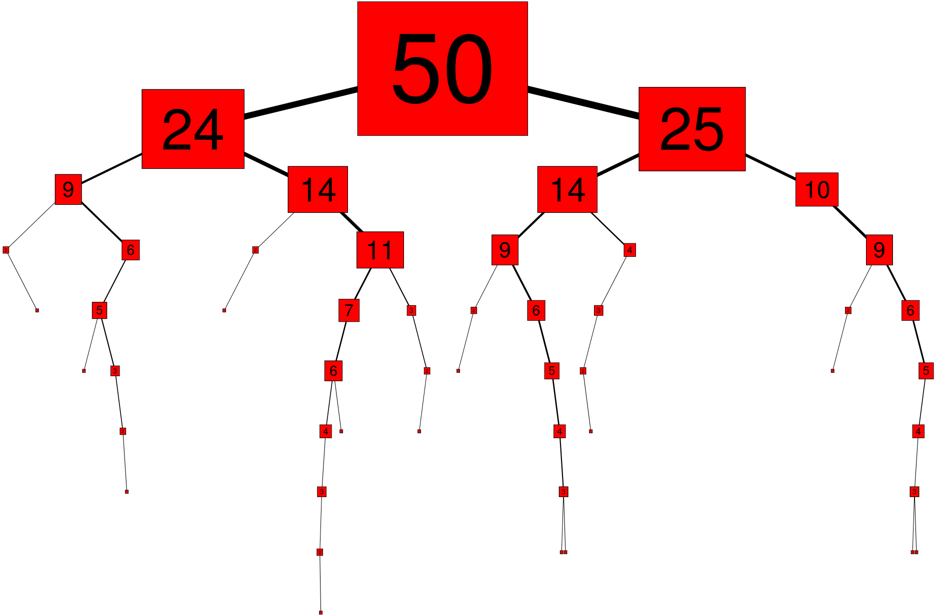
4. Actually, we will prove sth slightly stronger:

$$\mathbf{P}\left[\bigcap_{i=1}^{n} \{H_i \leq C \log n\}\right] \geq 1 - n^{-1}.$$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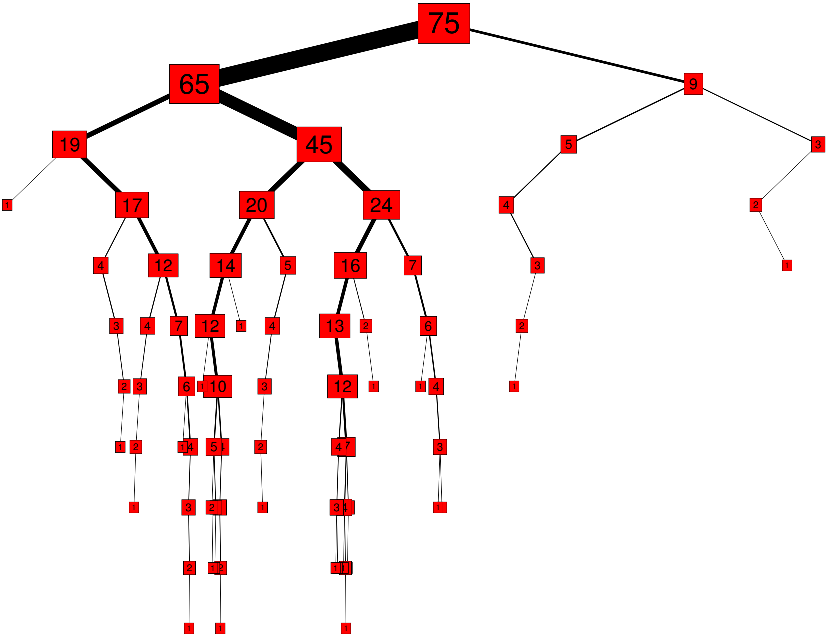
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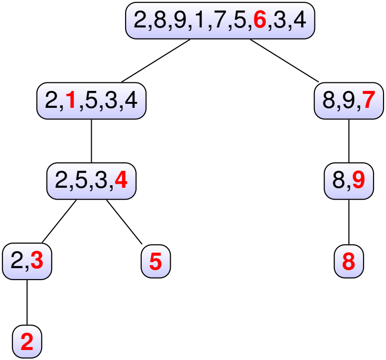

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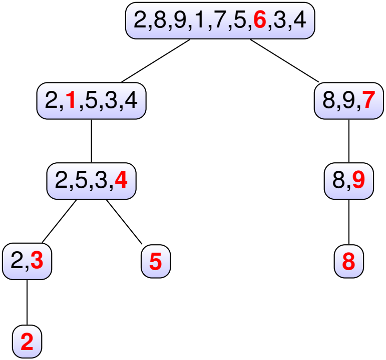


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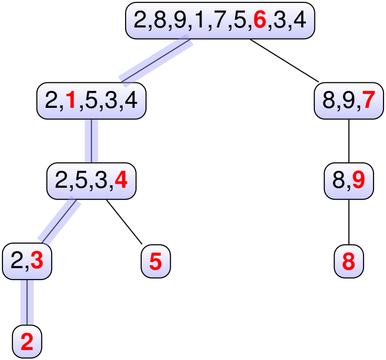


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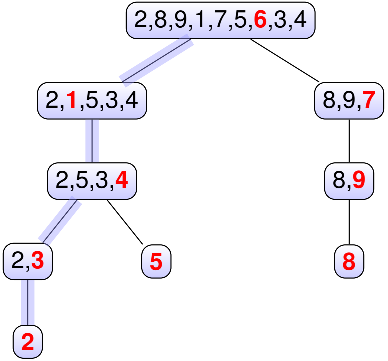


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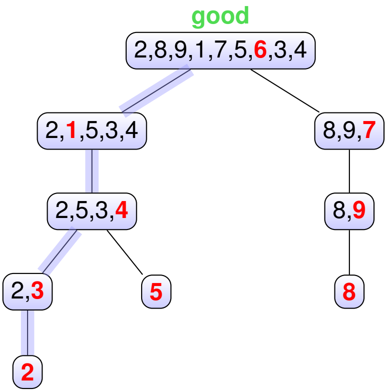


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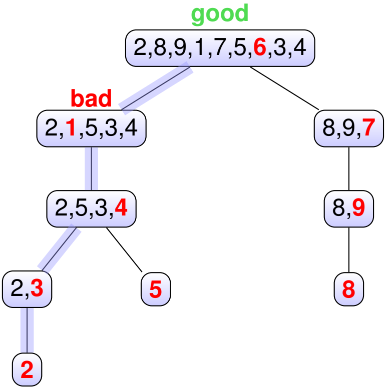


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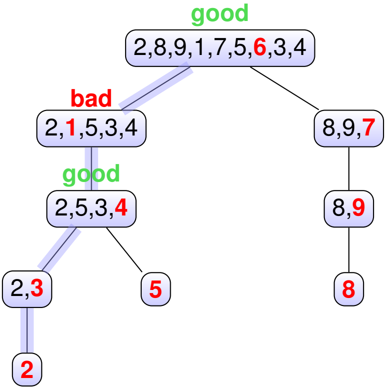


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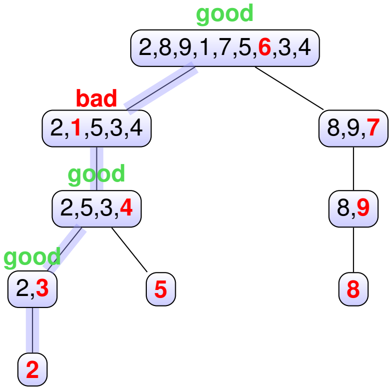


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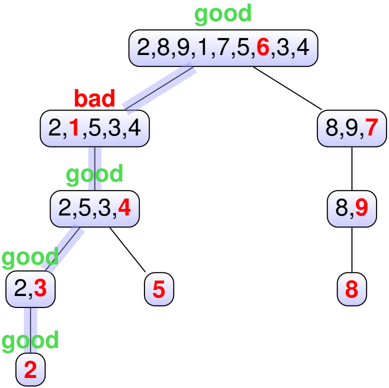


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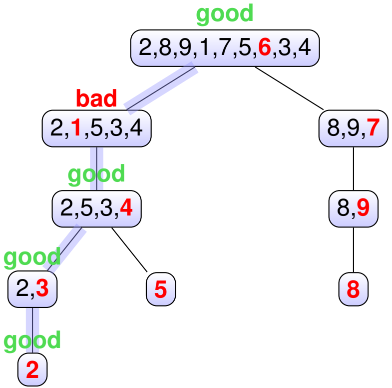


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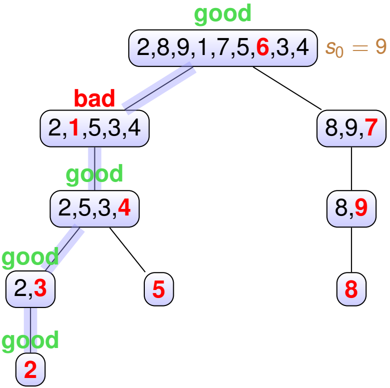


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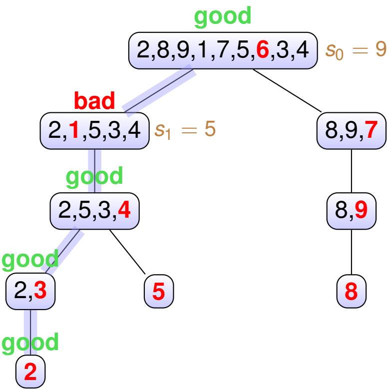


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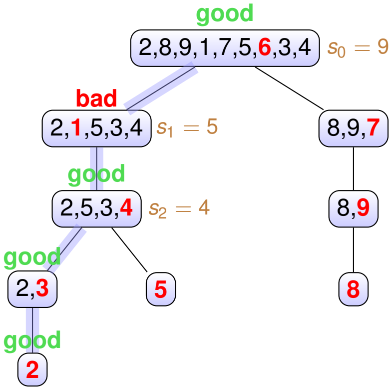


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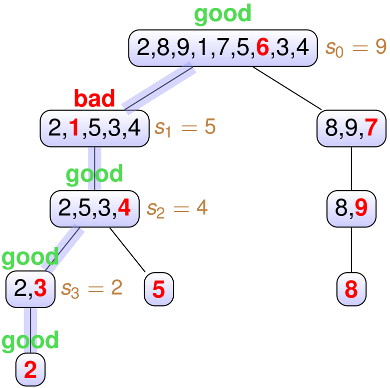


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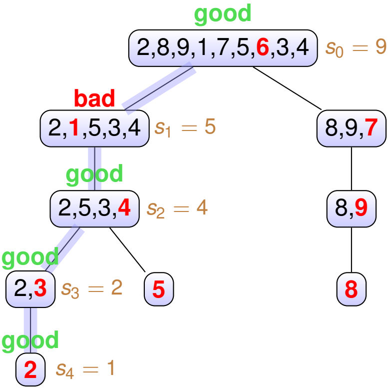


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Let us now upper bound the probability that this "bad event" happens!

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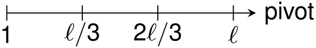

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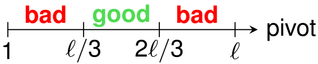

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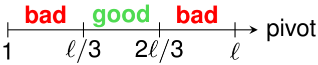


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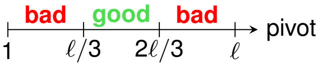


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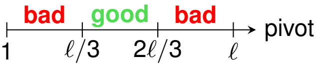


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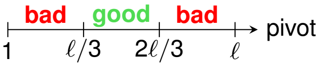


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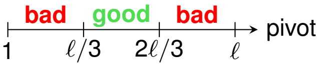


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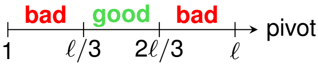


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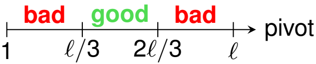


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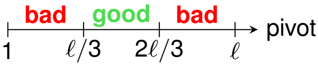


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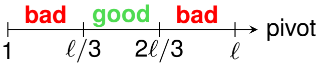


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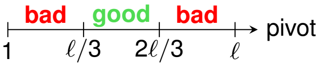


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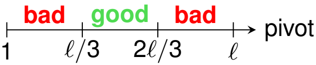


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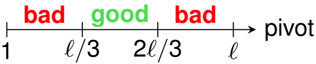


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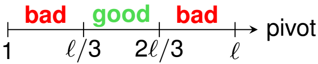


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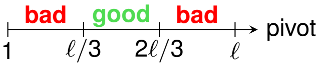


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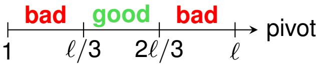


- $\mathbf{P}[X_j = 1 \mid X_0 = x_0, \ldots, X_{j-1} = x_{j-1}] \leq \frac{2}{3}$
- $X := \sum_{j=0}^{24 \log n - 1} X_j$ satisfies relaxed independence assumption (Lecture 2)

We can now apply a **Chernoff Bound**! (We use the "nice" version.)

- We have $\mathbf{E}[X] \leq (2/3) \cdot 24 \log n = 16 \log n$
- Then, by the "nicer" Chernoff Bounds

$\mathbf{P}[X \geq \mathbf{E}[X] + t] \leq e^{-2t^2/n}$

$$\mathbf{P}[X > 21 \log n] \leq \mathbf{P}[X > \mathbf{E}[X] + 5 \log n] \leq e^{-2(5 \log n)^2/(24 \log n)} \leq n^{-2}.$$

- Hence $P$ has more than $24 \log n$ nodes with probability at most $n^{-2}$.
- As there are in total $n$ paths, by the union bound, the probability that at least one of them has more than $24 \log n$ nodes is at most $n^{-1}$.
- This implies $\mathbf{P}\left[\bigcap_{i=1}^{n} \{H_i \leq 24 \log n\}\right] \geq 1 - n^{-1}$, as needed. □

# Randomised QuickSort: Final Remarks

- Well-known: any comparison-based sorting algorithm needs $\Omega(n \log n)$
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**Exercise:** *[Ex 2-3.6]* Our upper bound of $O(n \log n)$ whp also immediately implies a $O(n \log n)$ bound on the expected number of comparisons!

- It is possible to deterministically find the best pivot element that divides the array into two subarrays of the same size.
- The latter requires to compute the median of the array in linear time, which is not easy...
- The presented randomised algorithm for QUICKSORT is much easier to implement!

# Outline

Application 2: Randomised QuickSort

Extensions of Chernoff Bounds

Applications of Method of Bounded Differences

# Hoeffding's Extension

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Let $X$ be a random variable with mean 0 such that $a \leq X \leq b$. Then for all $\lambda \in \mathbb{R}$,

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We omit the proof of this lemma!

# Hoeffding Bounds

**Hoeffding's Inequality**

Let $X_1, \ldots, X_n$ be independent random variables with mean $\mu_i$ such that $a_i \leq X_i \leq b_i$. Let $X = X_1 + \ldots + X_n$, and let $\mu = \mathbf{E}[X] = \sum_{i=1}^n \mu_i$. Then for any $t > 0$,

$$\mathbf{P}[X \geq \mu + t] \leq \exp\left(-\frac{2t^2}{\sum_{i=1}^n (b_i - a_i)^2}\right),$$

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Proof Outline (skipped):

- Let $X_i' = X_i - \mu_i$ and $X' = X_1' + \ldots + X_n'$, then $\mathbf{P}[X \geq \mu + t] = \mathbf{P}[X' \geq t]$
- $\mathbf{P}[X' \geq t] \leq e^{-\lambda t} \prod_{i=1}^n \mathbf{E}\left[e^{\lambda X_i'}\right] \leq \exp\left[-\lambda t + \frac{\lambda^2}{8} \sum_{i=1}^n (b_i - a_i)^2\right]$
- Choose $\lambda = \frac{4t}{\sum_{i=1}^n (b_i - a_i)^2}$ to get the result.

This is not "magic" – you just need to optimise $\lambda$!

# Method of Bounded Differences

Framework

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In all those cases (and more) we can easily prove concentration of $f(X_1, \ldots, X_n)$ around its mean by the so-called **Method of Bounded Differences**.

# Method of Bounded Differences

A function $f$ is called Lipschitz with parameters $\mathbf{c} = (c_1, \ldots, c_n)$ if for all $i = 1, 2, \ldots, n$,

$$|f(x_1, x_2, \ldots, x_{i-1}, \boldsymbol{x_i}, x_{i+1}, \ldots, x_n) - f(x_1, x_2, \ldots, x_{i-1}, \boldsymbol{\widetilde{x}_i}, x_{i+1}, \ldots, x_n)| \leq c_i,$$

where $x_i$ and $\widetilde{x}_i$ are in the domain of the $i$-th coordinate.

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Let $X_1, \ldots, X_n$ be independent random variables. Let $f$ be Lipschitz with parameters $\mathbf{c} = (c_1, \ldots, c_n)$. Let $X = f(X_1, \ldots, X_n)$. Then for any $t > 0$,

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- The proof is omitted here (it requires the concept of martingales).

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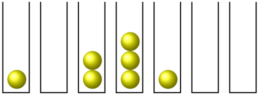
# Application 3: Balls into Bins (again...)



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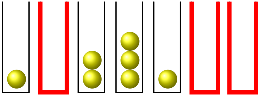

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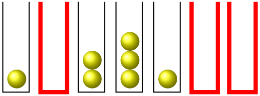

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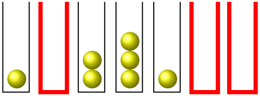

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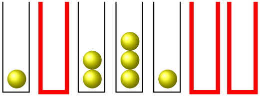

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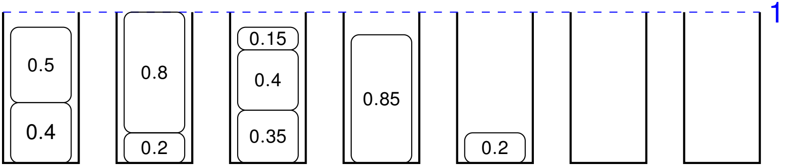
This is a decent bound, but for some values of $m$ it is far from tight and stronger bounds are possible through a refined analysis.

# Application 4: Bin Packing

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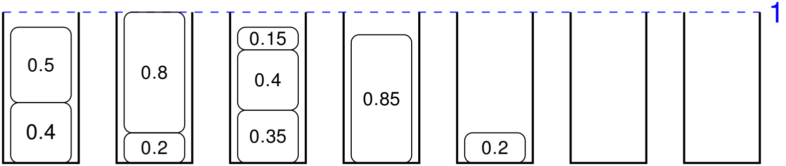


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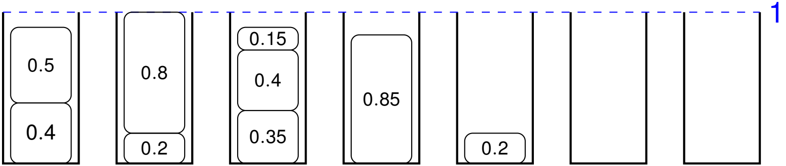


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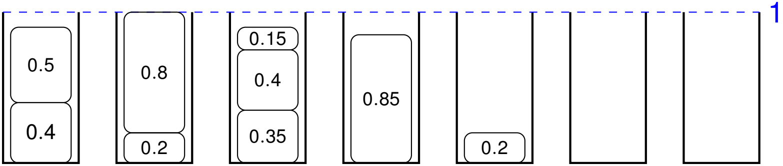


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- Therefore

$$\mathbf{P}\left[\,|B - \mathbf{E}\left[\,B\,\right]| \geq t\,\right] \leq 2 \cdot e^{-2t^2/n}.$$

This is a typical example where proving concentration is much easier than calculating (or estimating) the expectation!