Randomised Algorithms

Lecture 4: Markov Chains and Mixing Times

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UNIVERSITY OF CAMBRIDGE

Outline

Recap of Markov Chain Basics

Irreducibility, Periodicity and Convergence

Total Variation Distance and Mixing Times

Application 1: Markov Chain Monte Carlo

Application 2: Card Shuffling

Appendix: Remarks on Mixing Time (non-examin.)

Applications of Markov Chains in Computer Science

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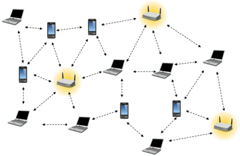
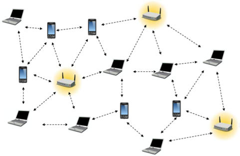


Broadcasting

Applications of Markov Chains in Computer Science



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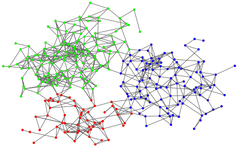
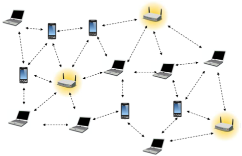


Clustering

Applications of Markov Chains in Computer Science



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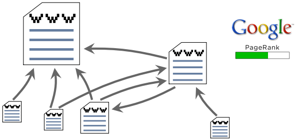

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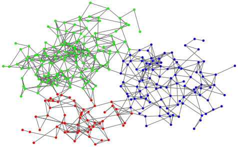

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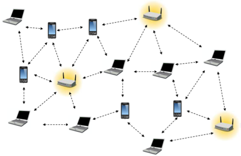

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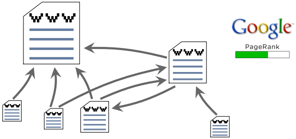

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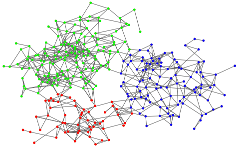

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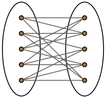

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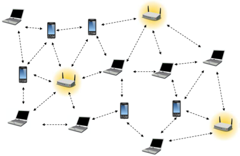

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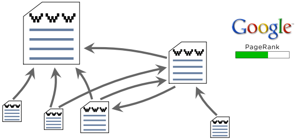

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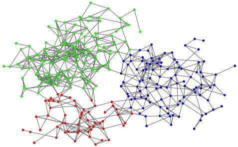

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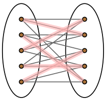

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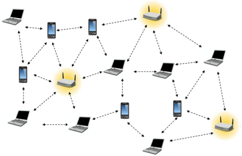

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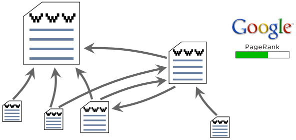

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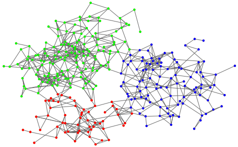

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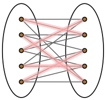

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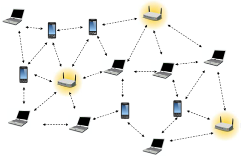
Sampling and Optimisation

Applications of Markov Chains in Computer Science



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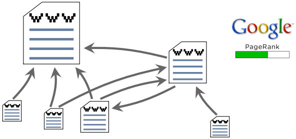

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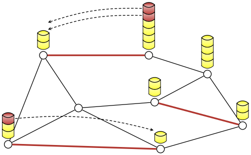
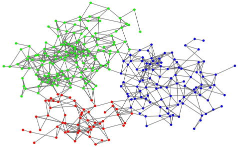


Load Balancing



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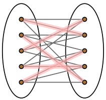

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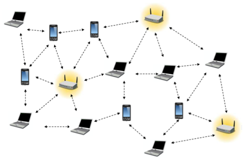

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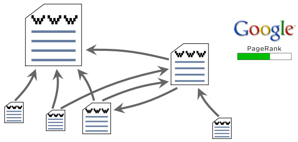

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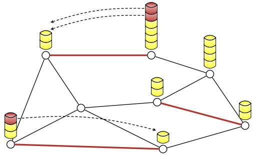

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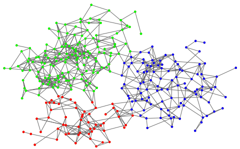

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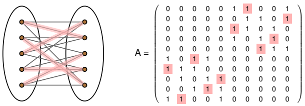

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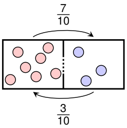

Particle Processes

Markov Chains

Markov Chain (Discrete Time and State, Time Homogeneous)

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$$:= P(x_t, x_{t+1}).$$

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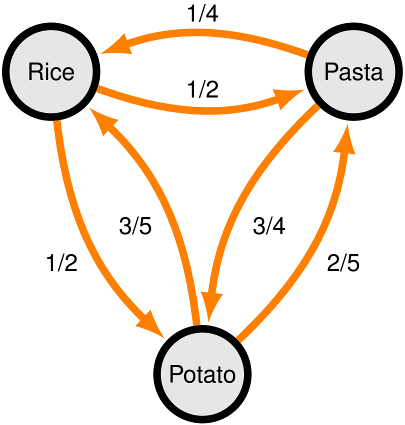
- For all $0 \leq t_1 < t_2, x \in \Omega$,

$$\mathbf{P}\left[X_{t_2} = x \right] = \sum_{y \in \Omega} \mathbf{P}\left[X_{t_2} = x \mid X_{t_1} = y \right] \cdot \mathbf{P}\left[X_{t_1} = y \right].$$

What does a Markov Chain Look Like?

Example: the carbohydrate served with lunch in the college cafeteria.



This has transition matrix:

$$P = \begin{array}{cc} & \begin{array}{ccc} \text{Rice} & \text{Pasta} & \text{Potato} \end{array} \\ & \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/4 & 0 & 3/4 \\ 3/5 & 2/5 & 0 \end{bmatrix} \begin{array}{l} \text{Rice} \\ \text{Pasta} \\ \text{Potato} \end{array} \end{array}$$

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$\Rightarrow$ can replace ρ by any (load) vector and view P as a balancing matrix!

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Irreducibility, Periodicity and Convergence

Total Variation Distance and Mixing Times

Application 1: Markov Chain Monte Carlo

Application 2: Card Shuffling

Appendix: Remarks on Mixing Time (non-examin.)

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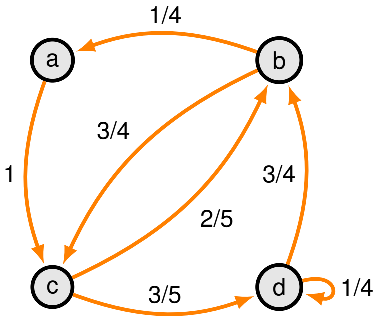

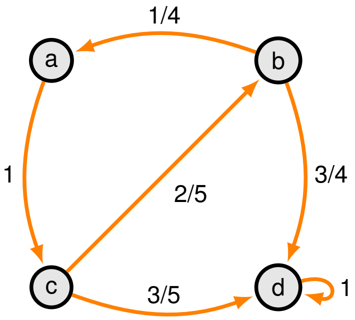

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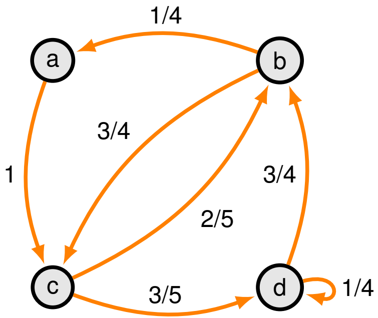

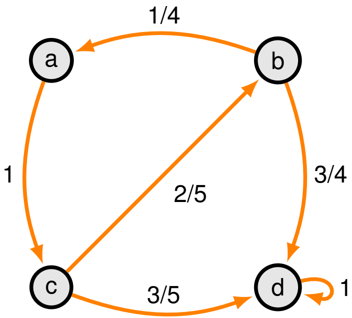

Question: Which of the two chains (if any) are irreducible?

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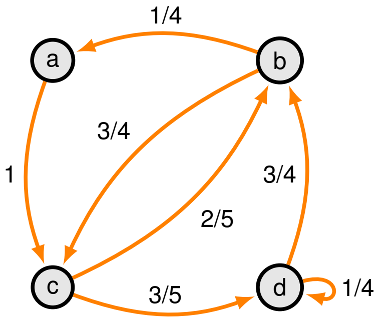
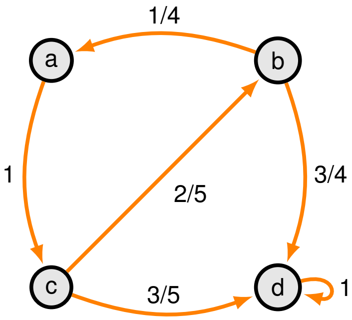


✓ irreducible



× not irreducible (thus reducible)

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College carbs example:

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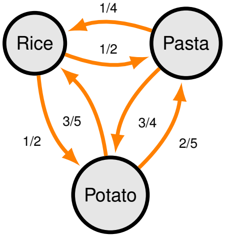

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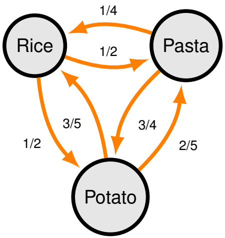

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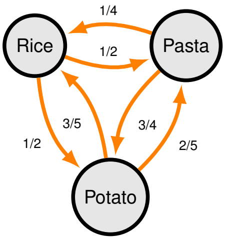

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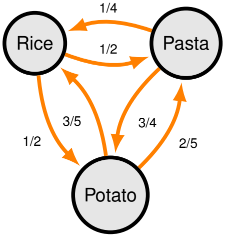

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Existence and Uniqueness of a Positive Stationary Distribution

Let P be finite, irreducible MC, then there exists a unique probability distribution π on Ω such that $\pi = \pi P$ and $\pi(x) = 1/h(x, x) > 0$, $\forall x \in \Omega$; $h(x, x)$ is the expected time for the MC starting in x to return to x.

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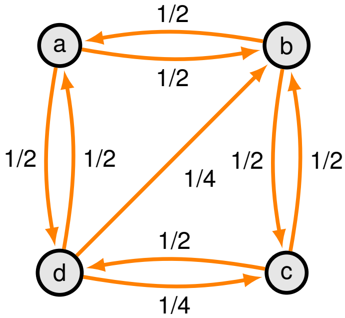

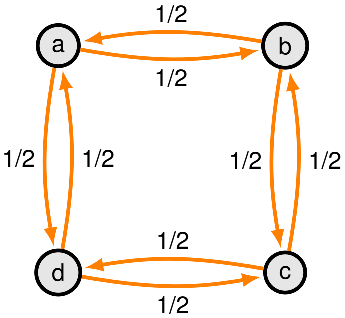

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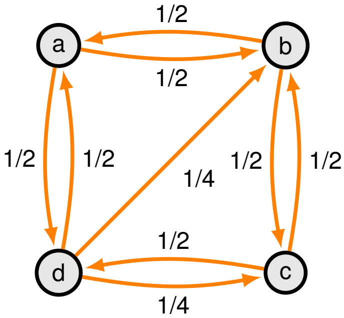

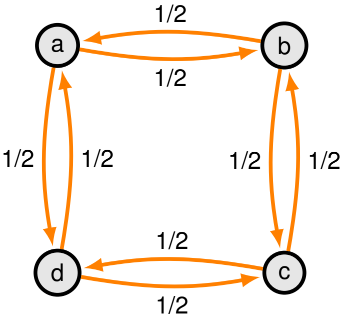

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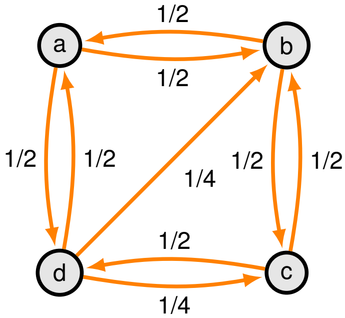

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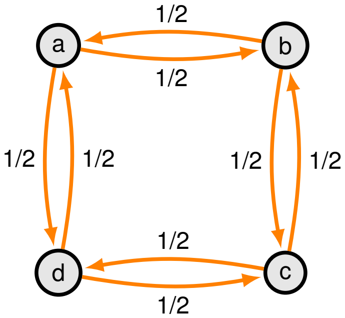

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- We will prove a quantitative version of the Convergence Theorem after introducing Spectral Graph Theory.

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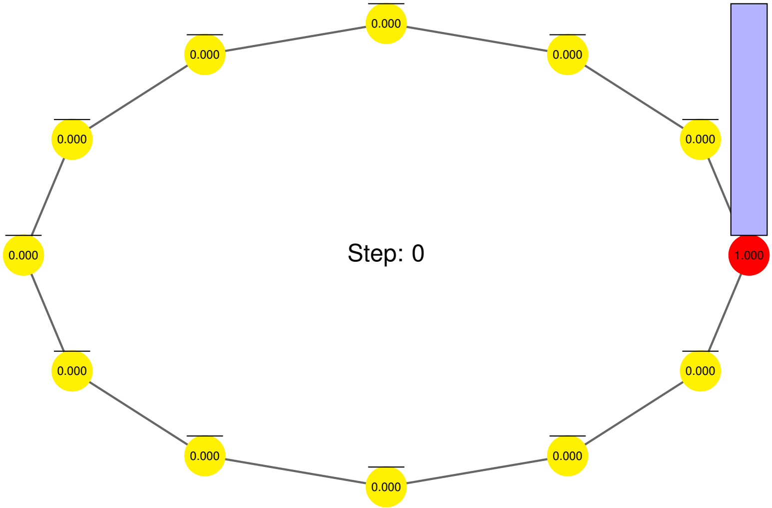

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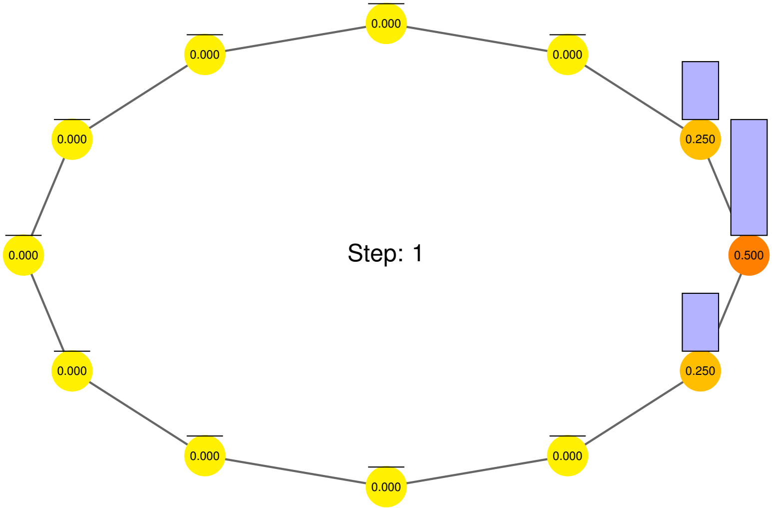

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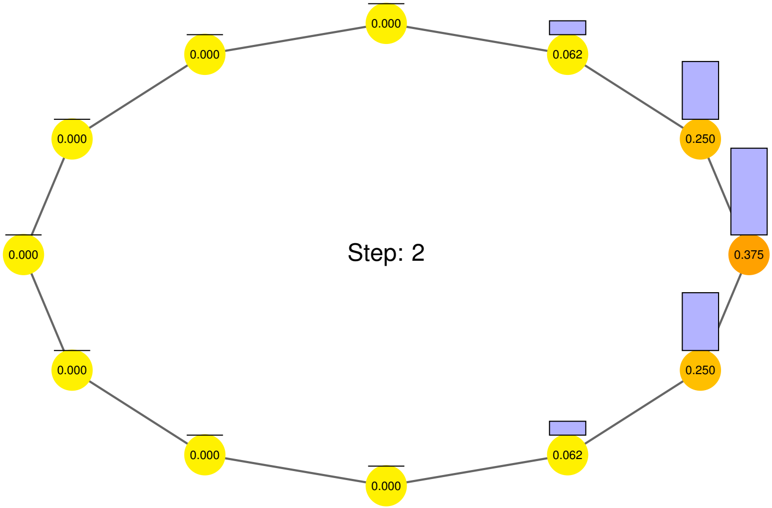

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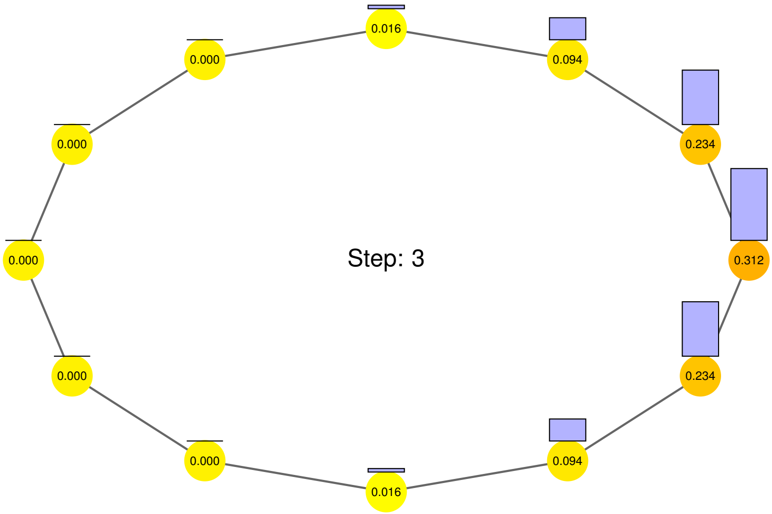

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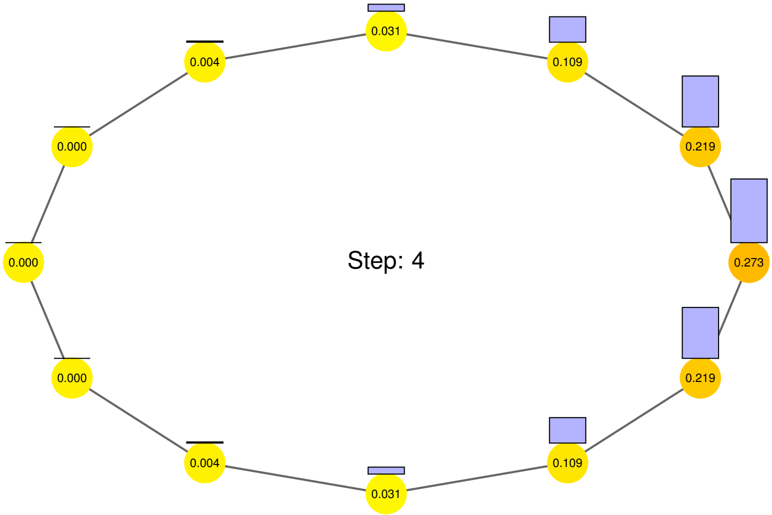

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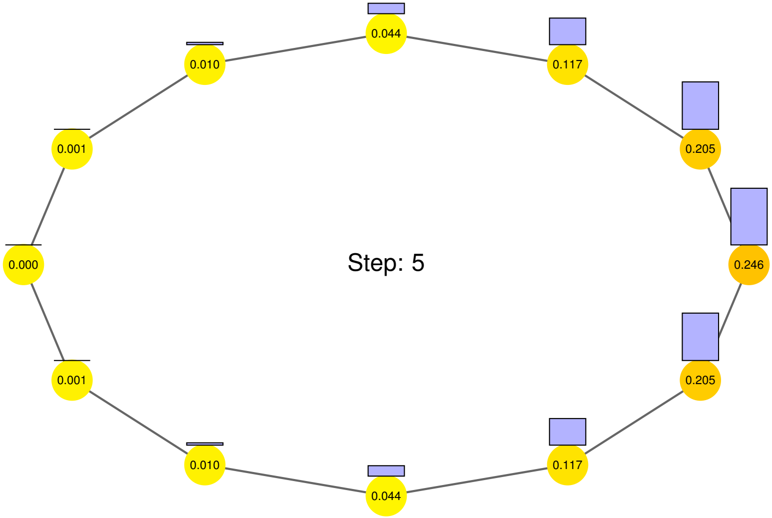

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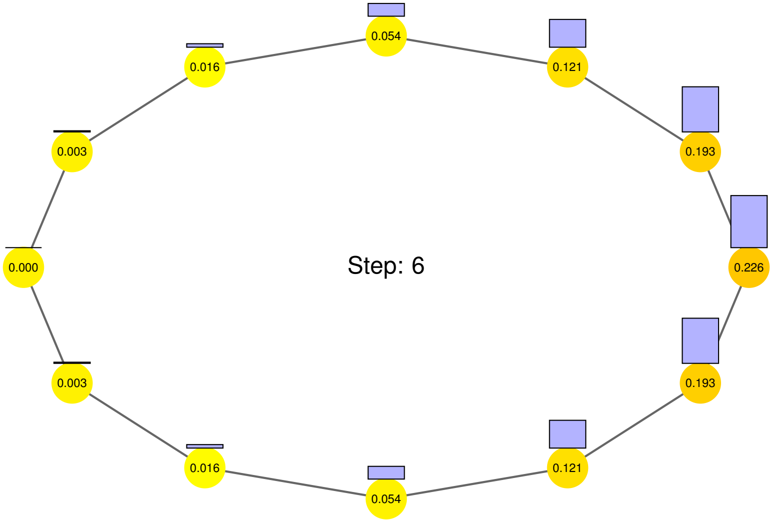

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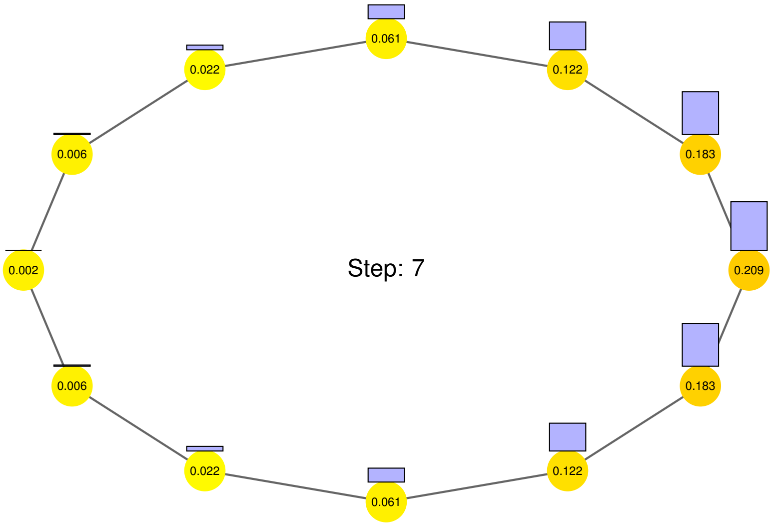

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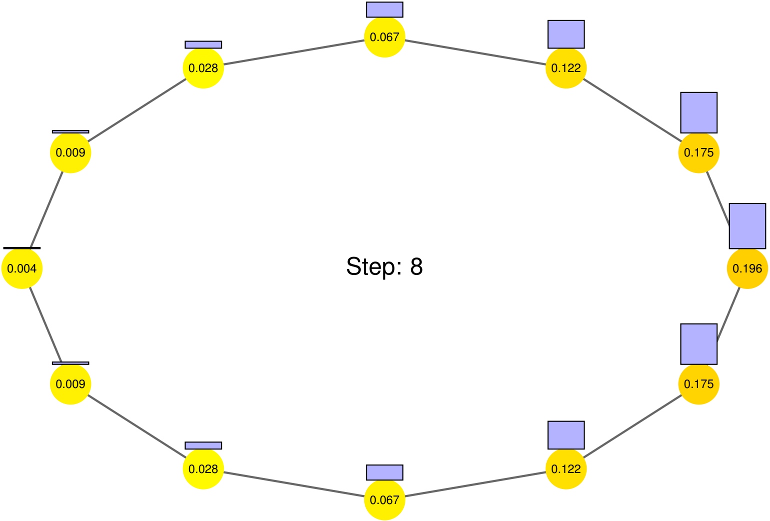

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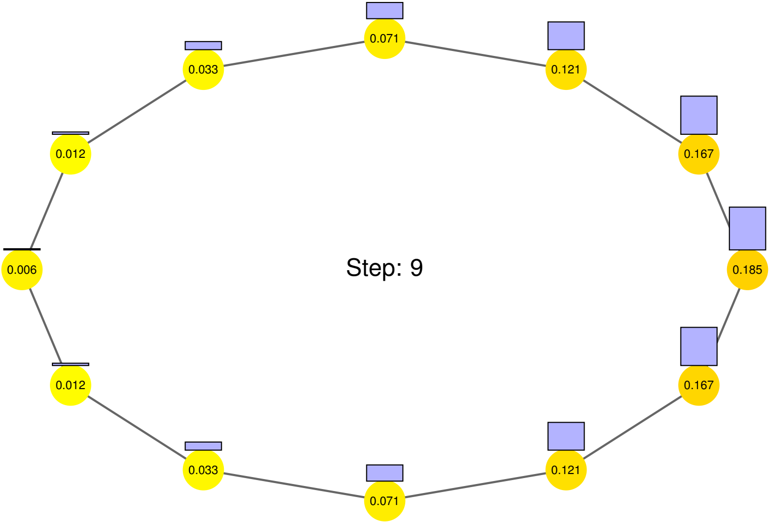

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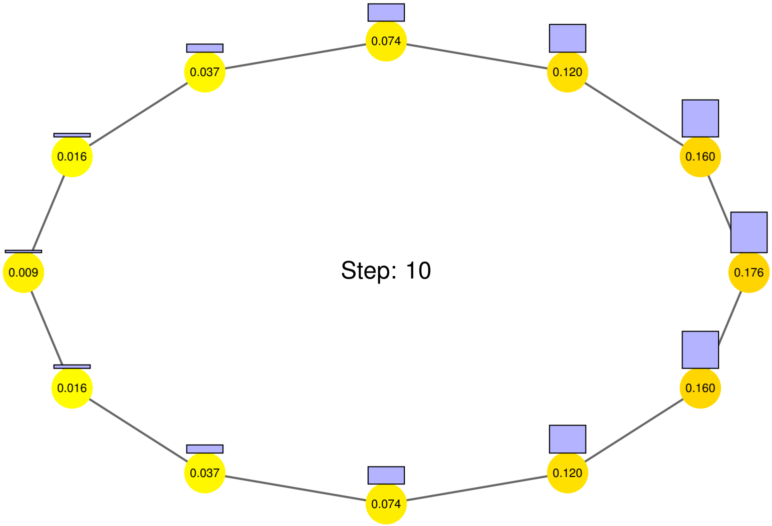

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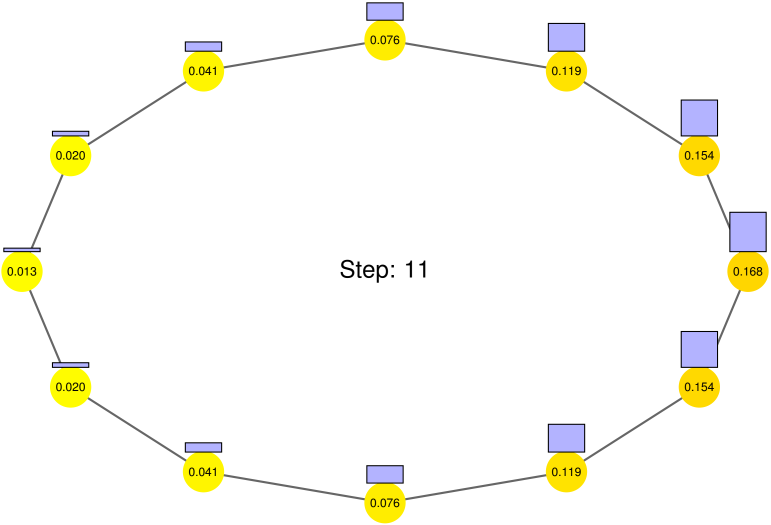

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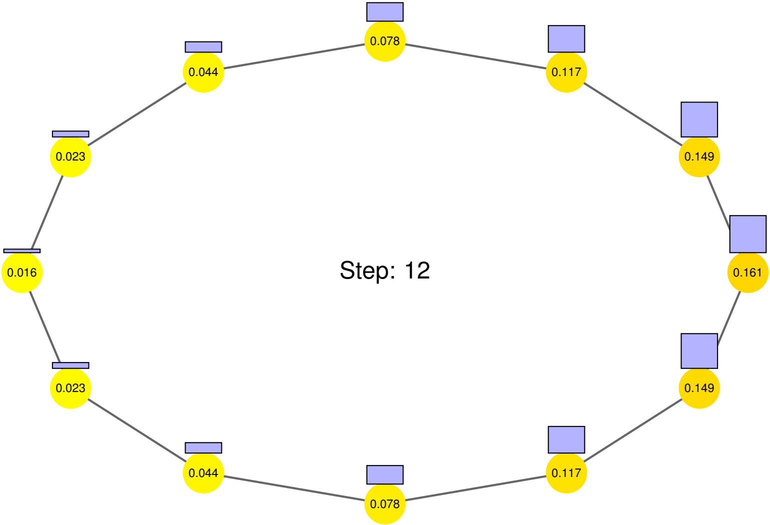

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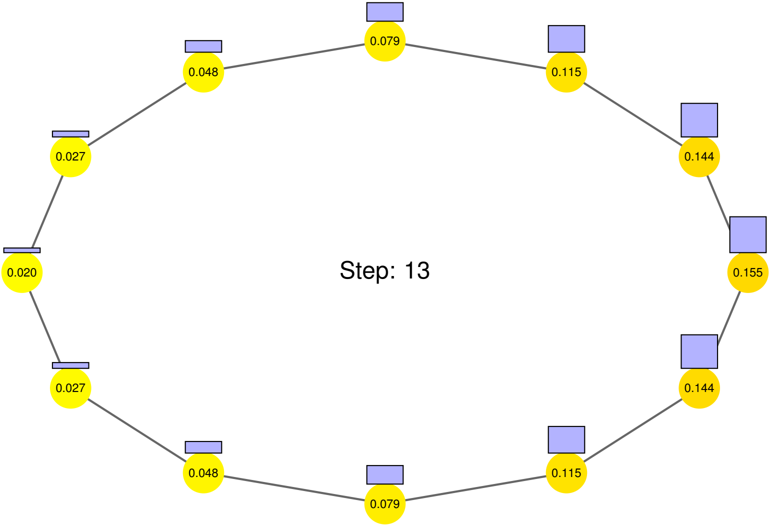

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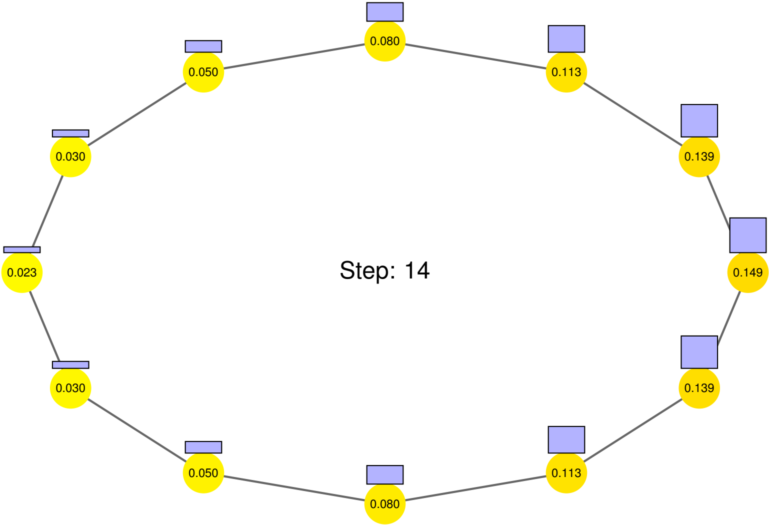

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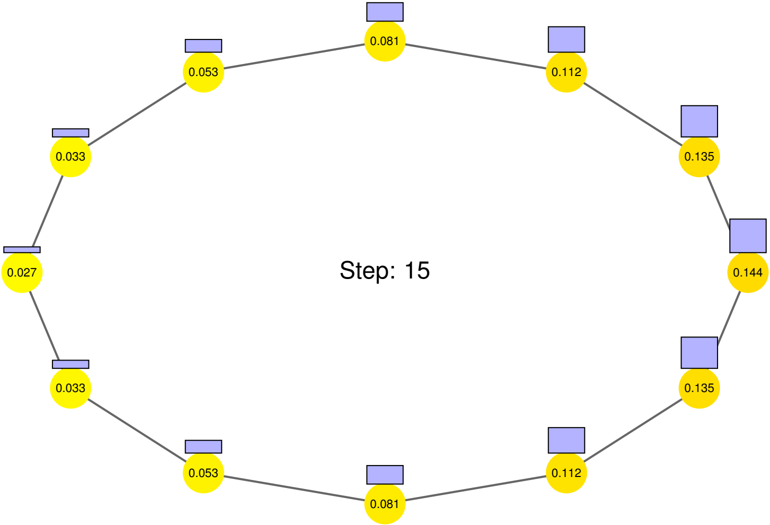

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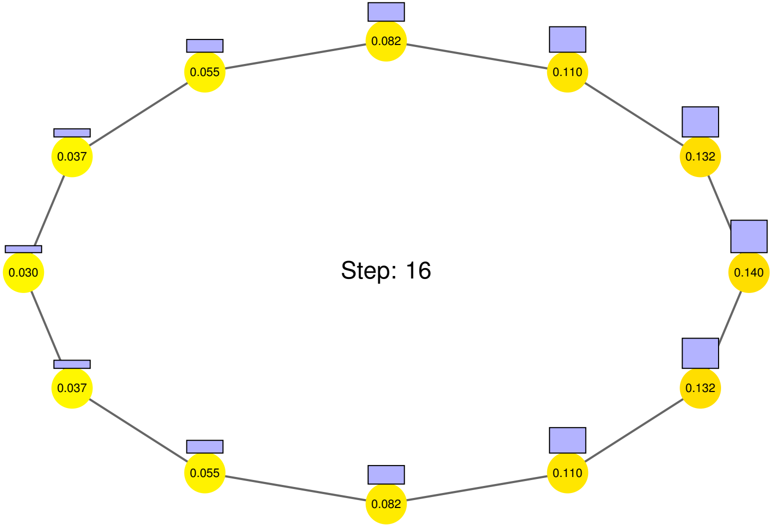

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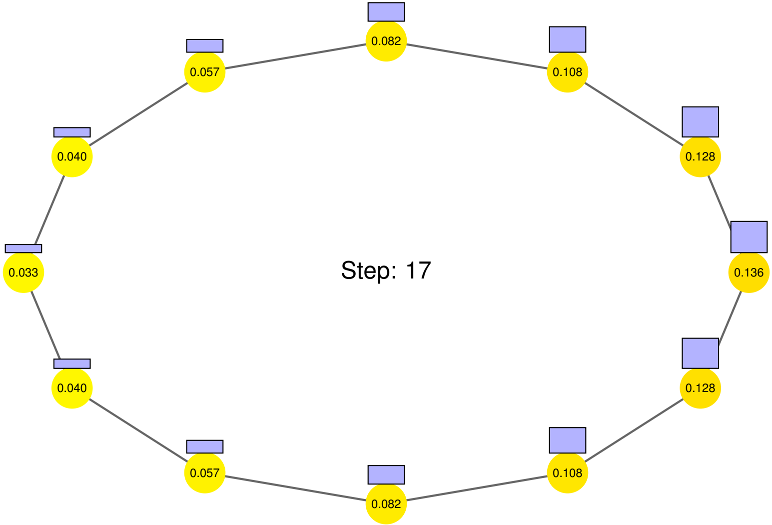

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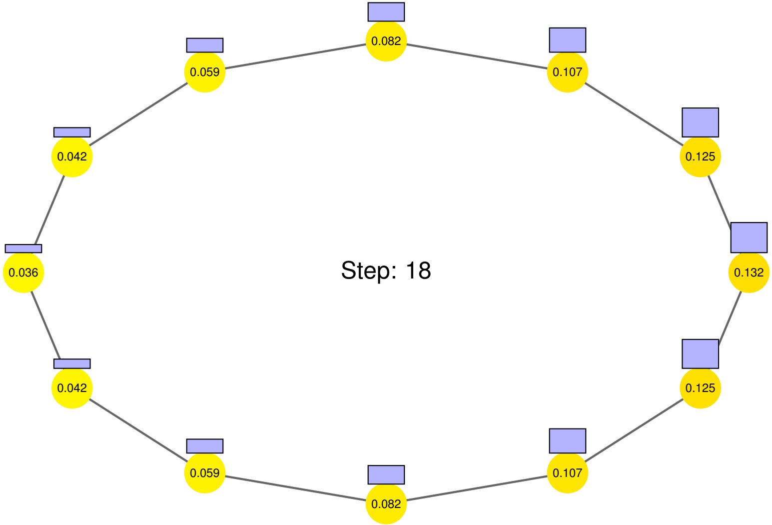

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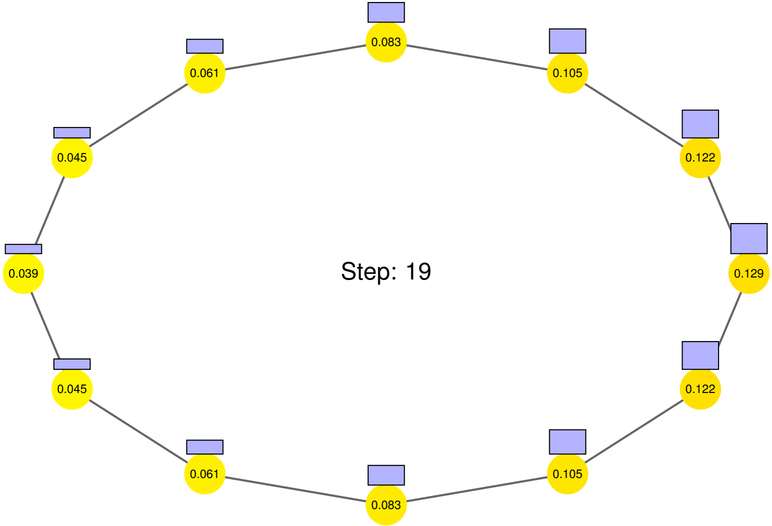

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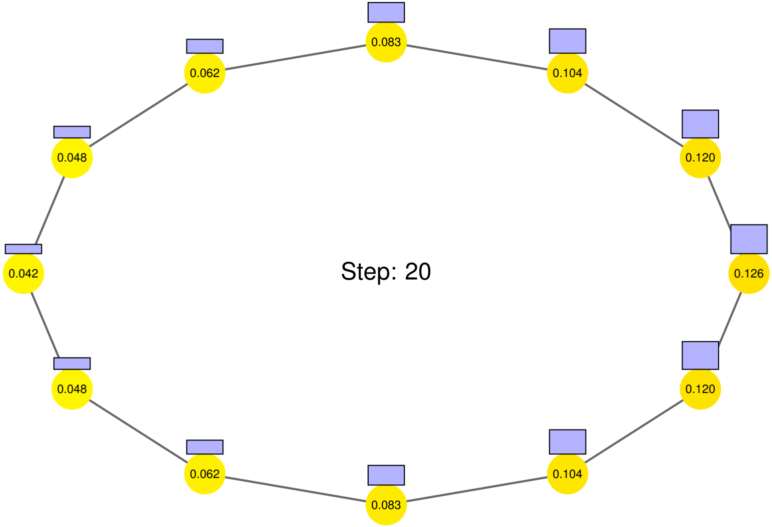

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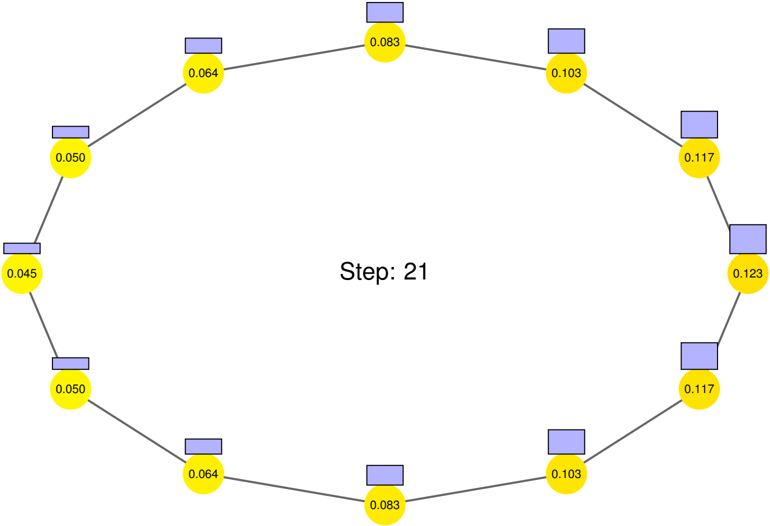

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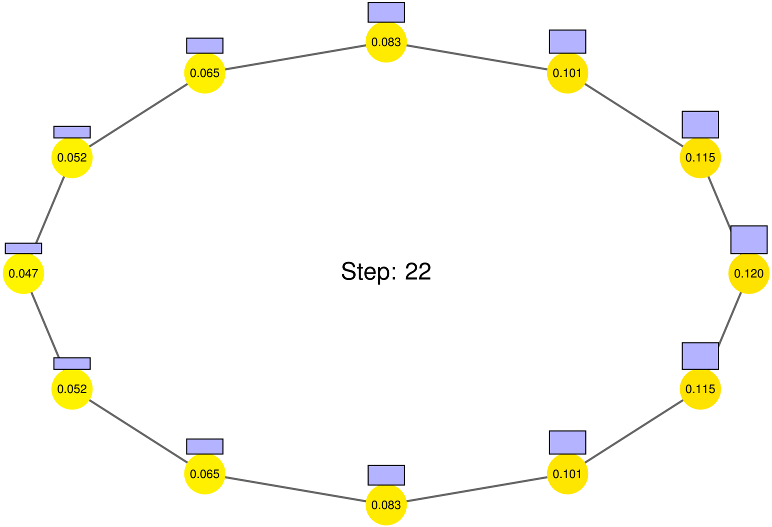

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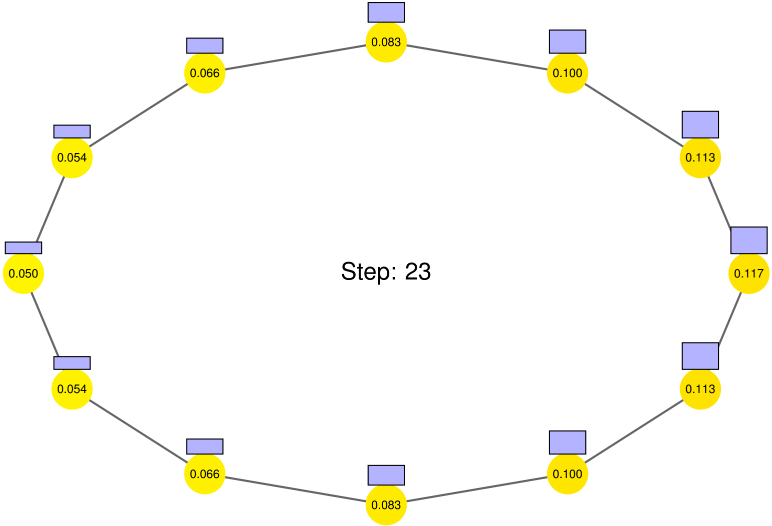

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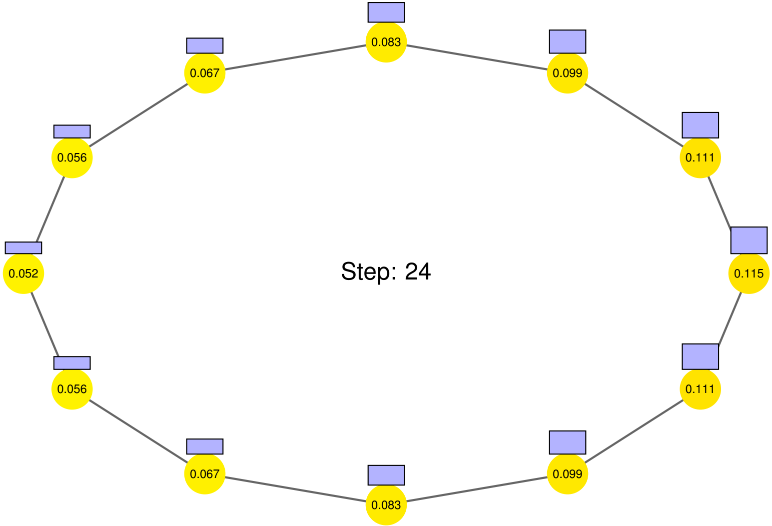

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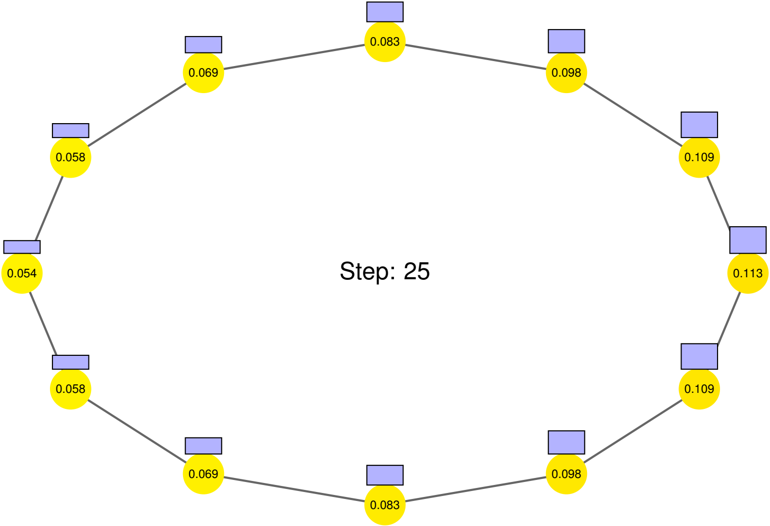

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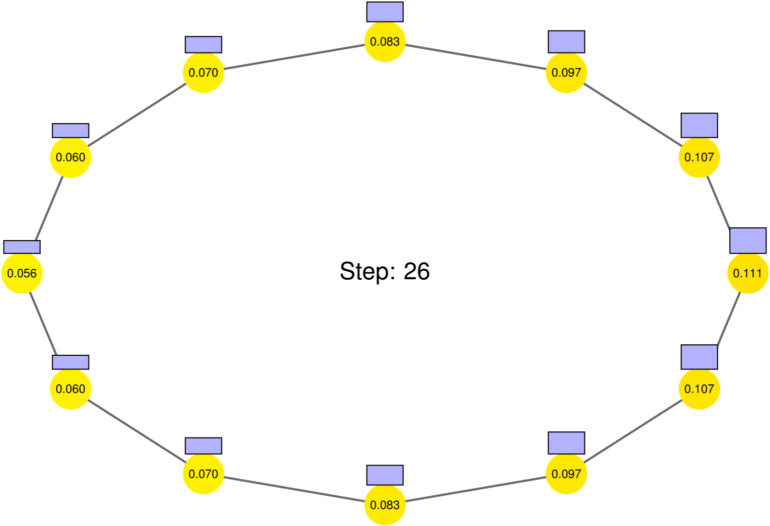

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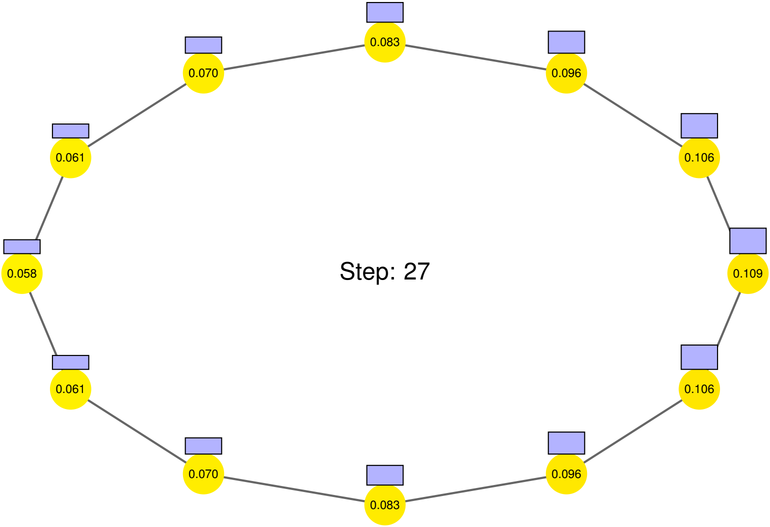

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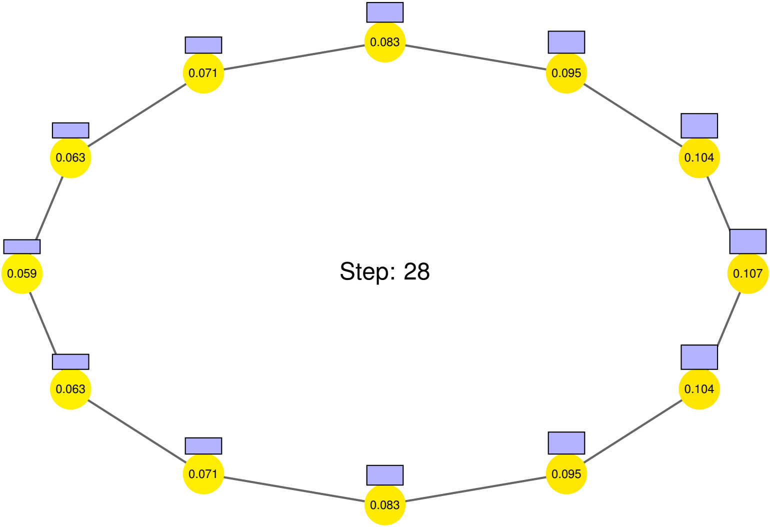

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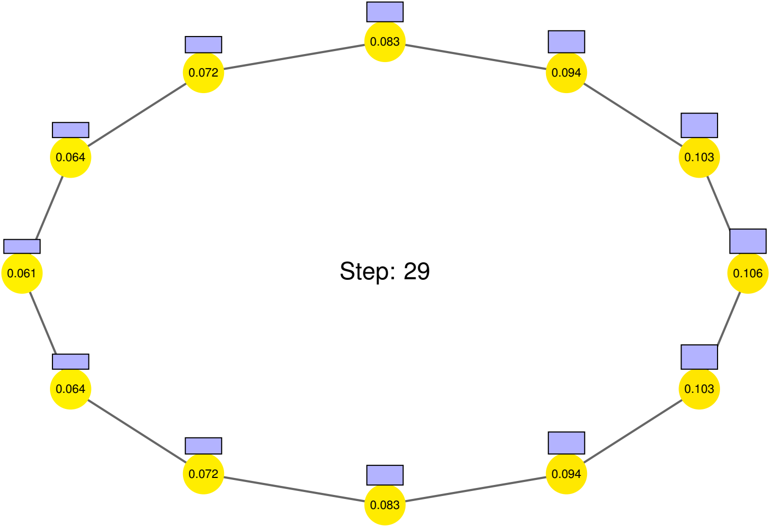

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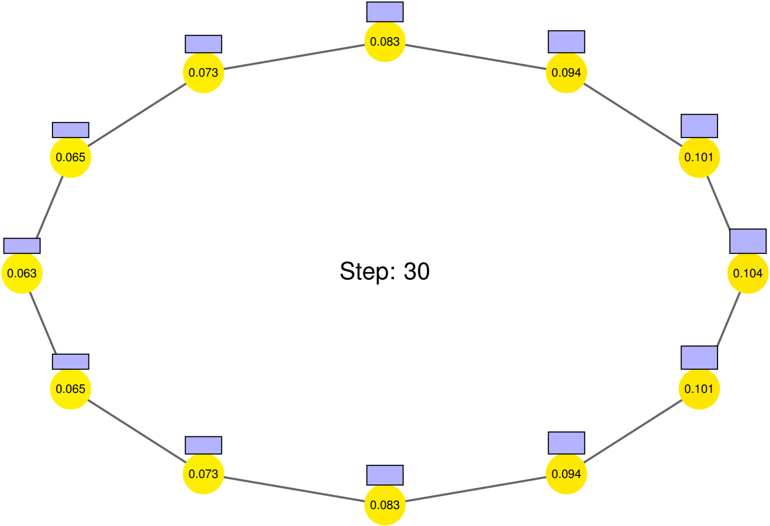

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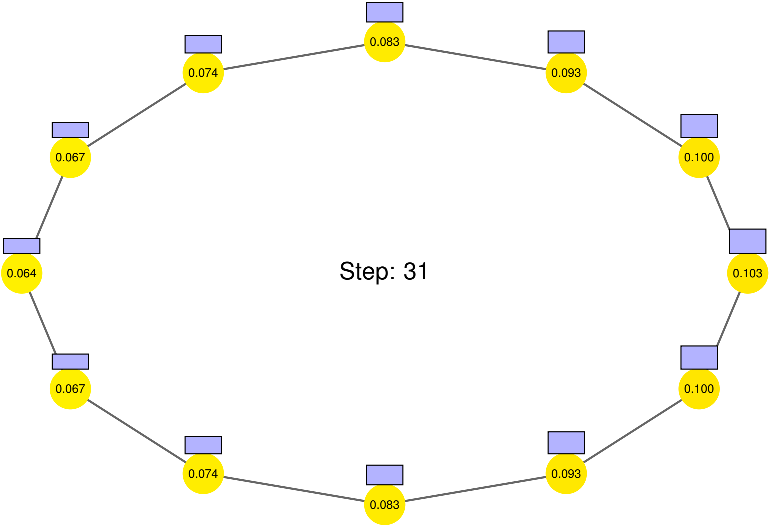

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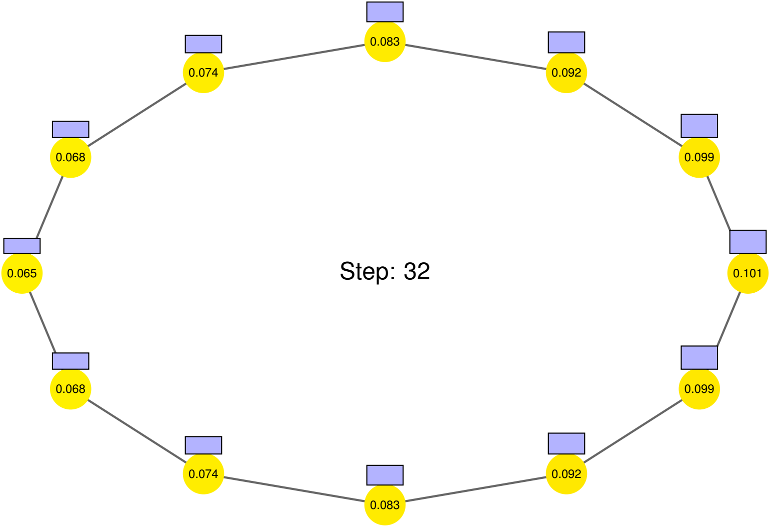

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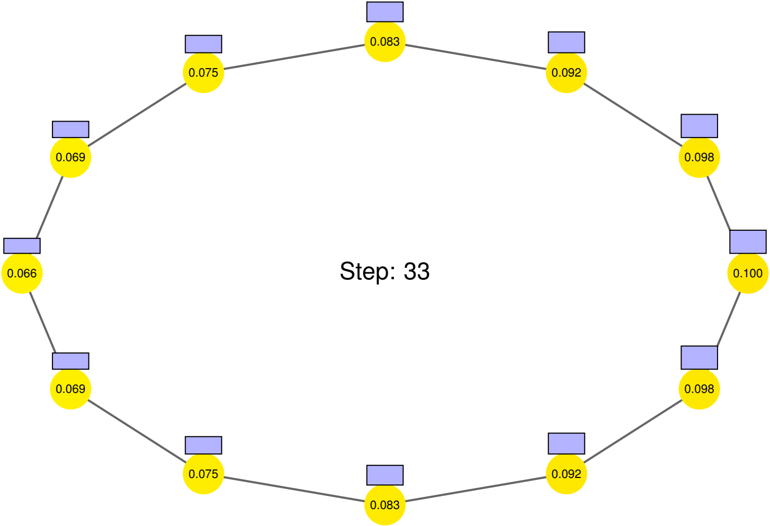

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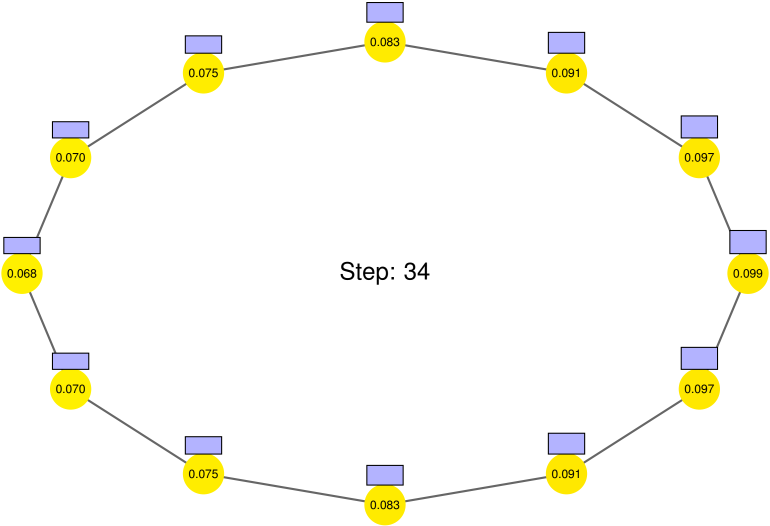

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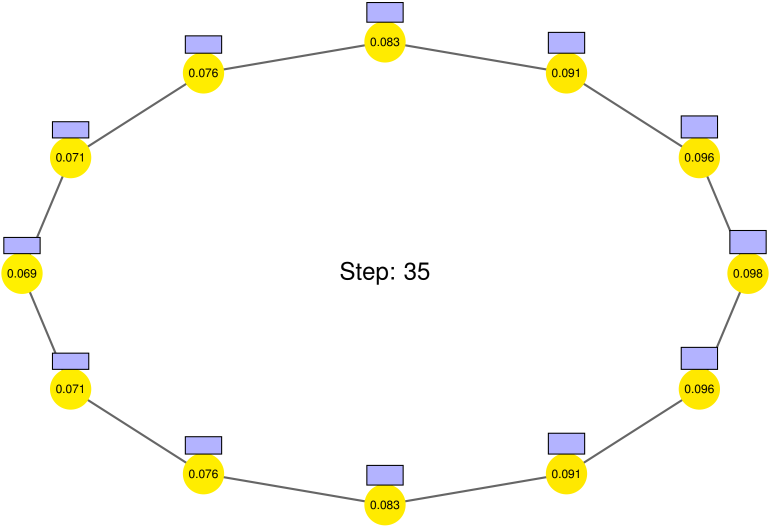

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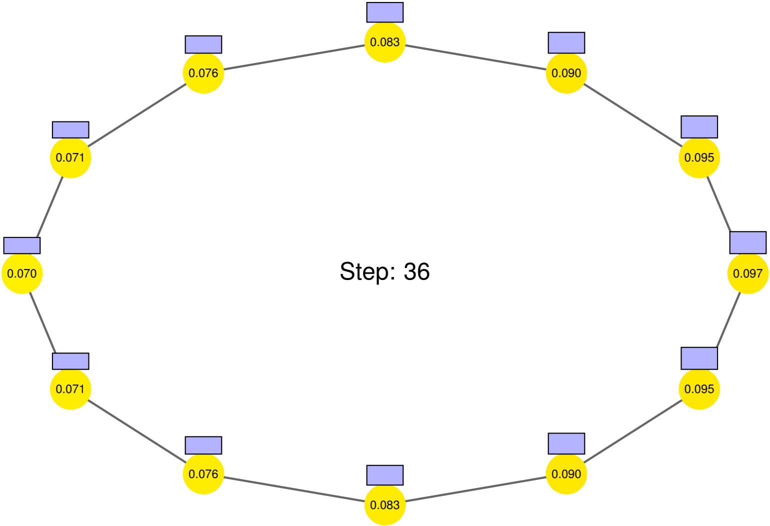

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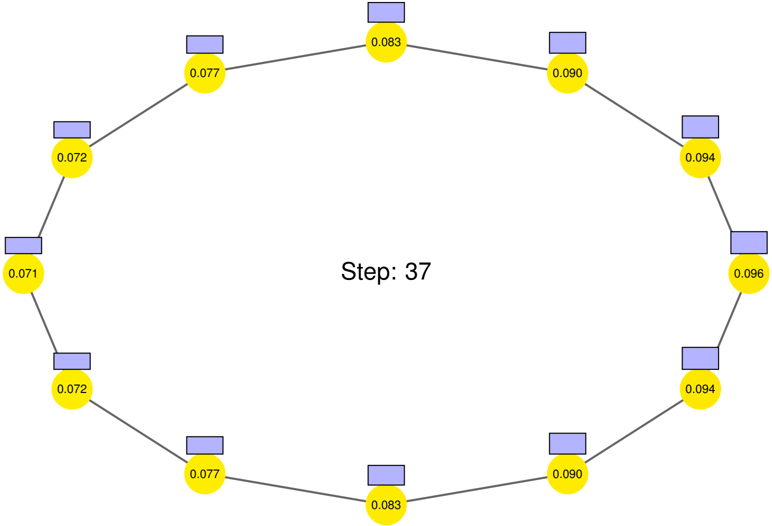

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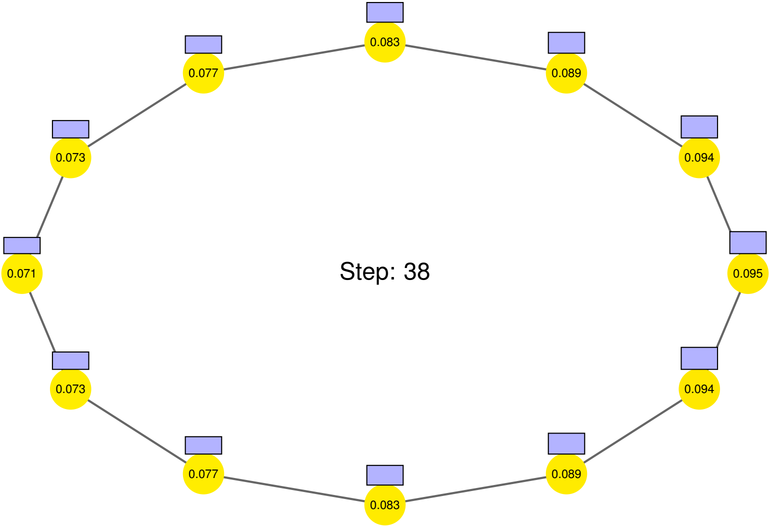

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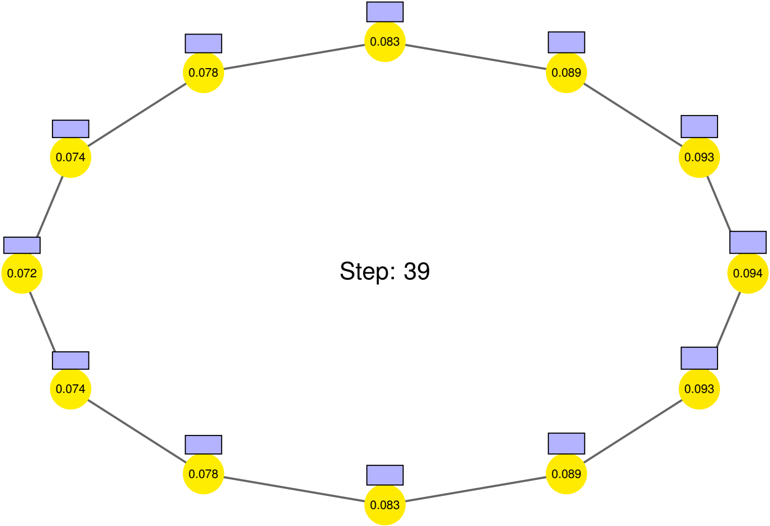

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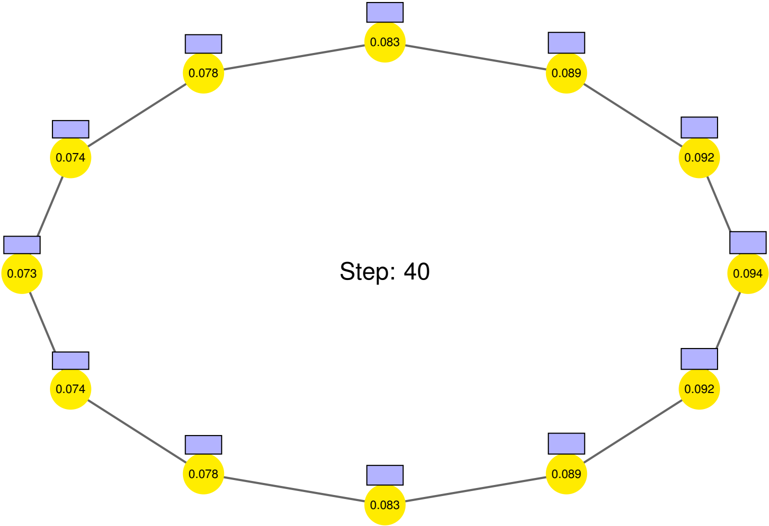

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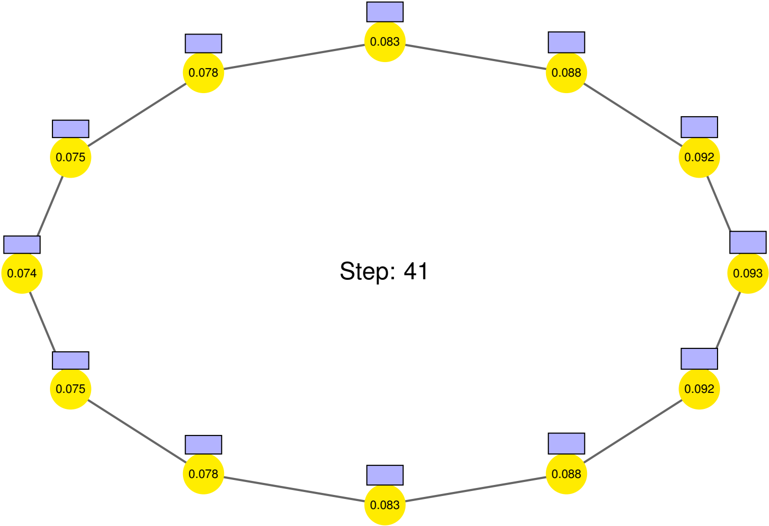

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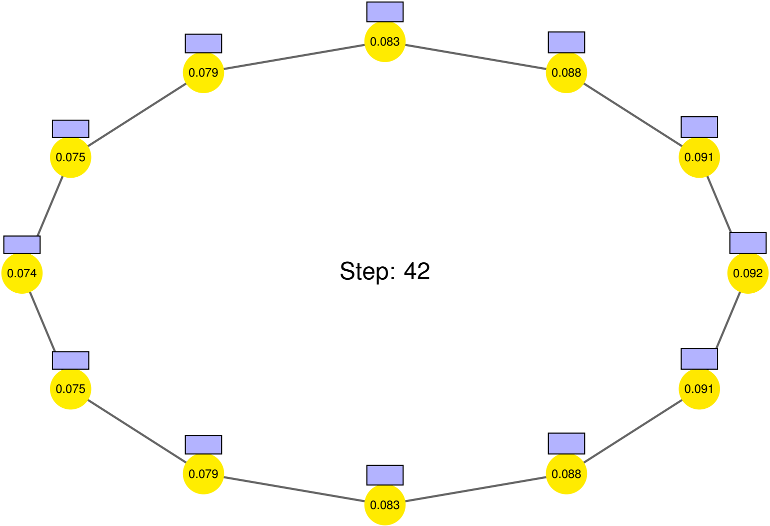

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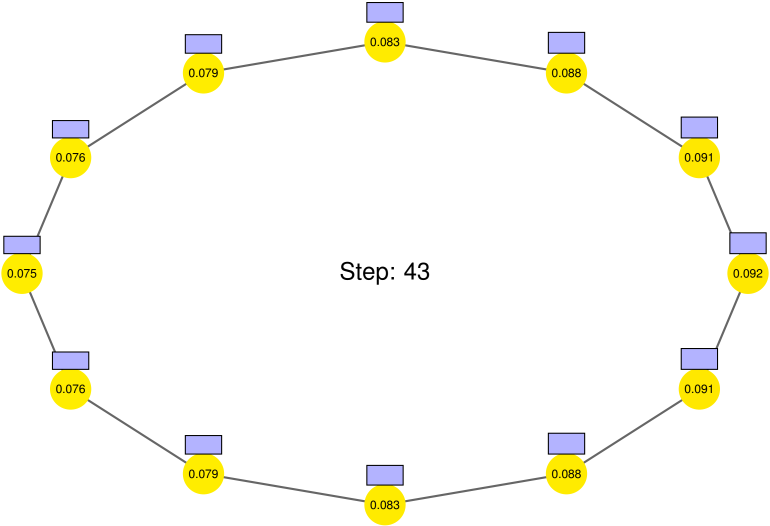

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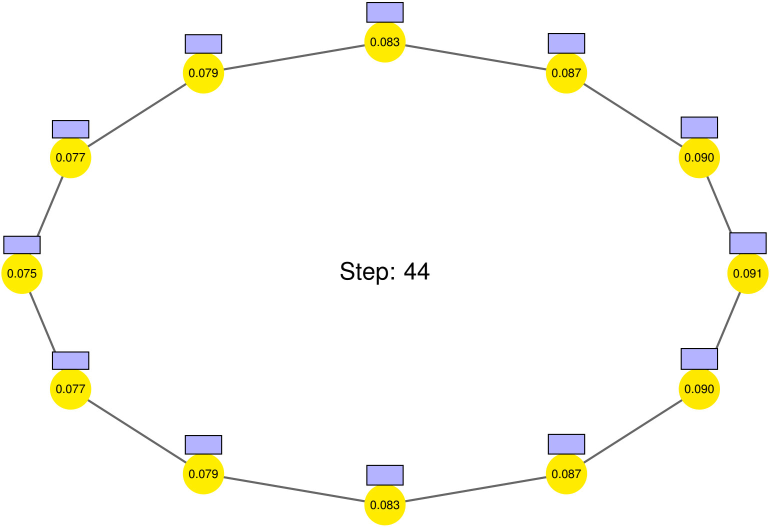

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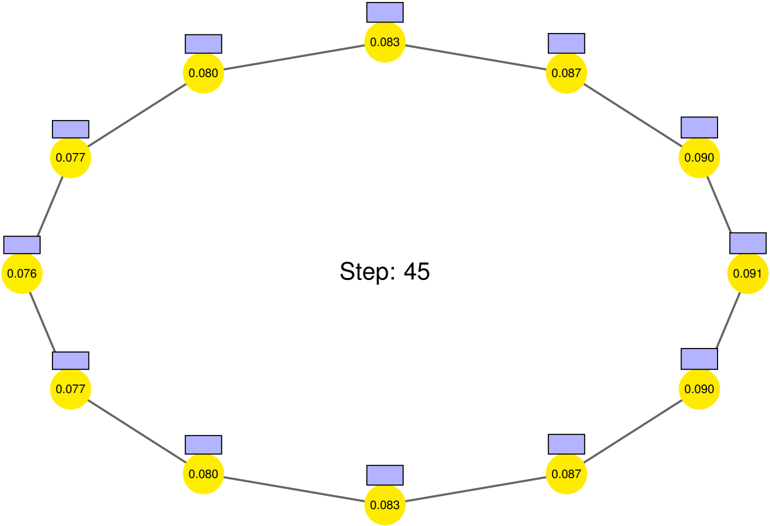

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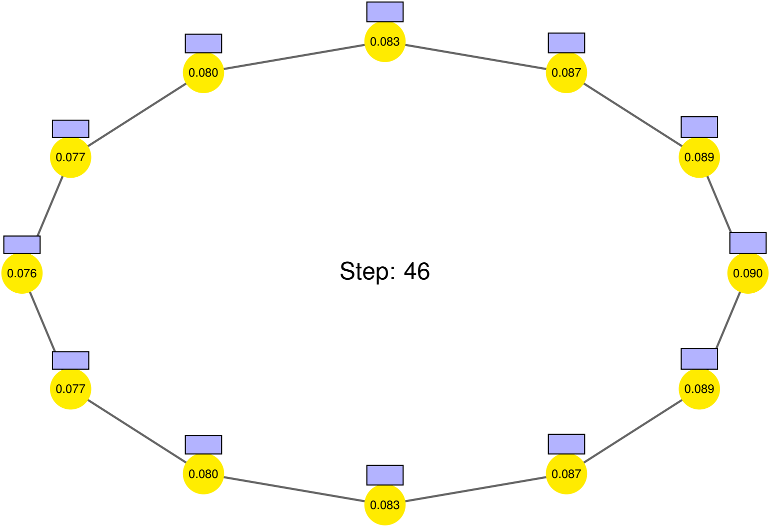

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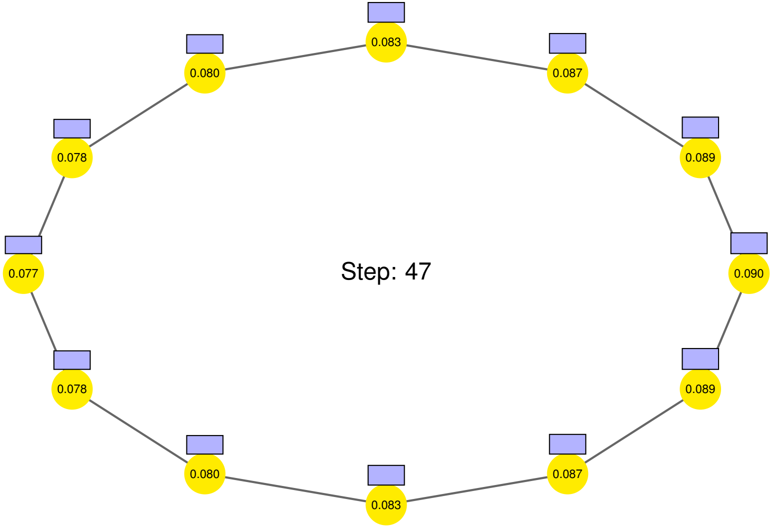

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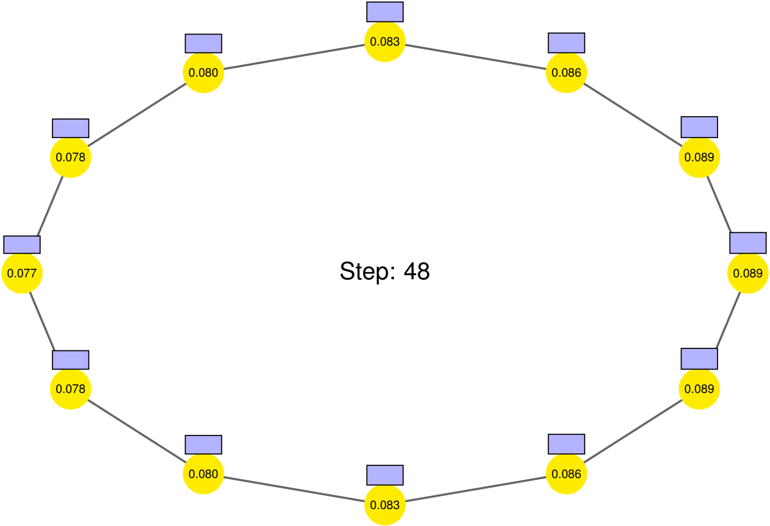

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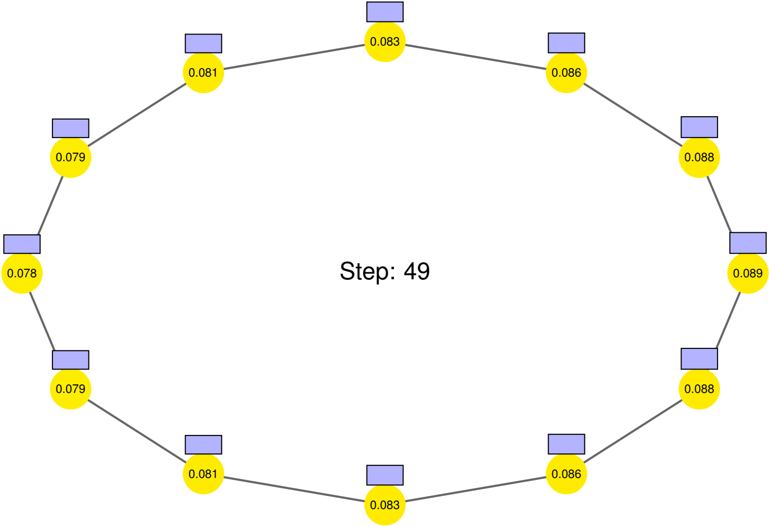

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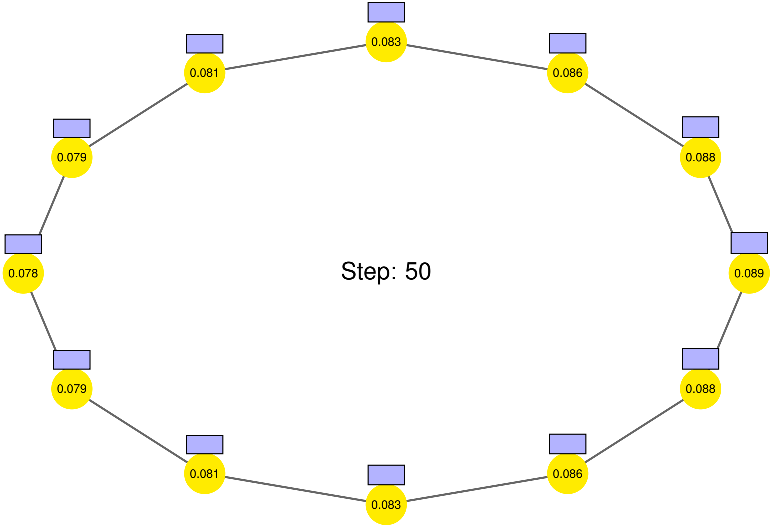

Outline

How Similar are Two Probability Measures?

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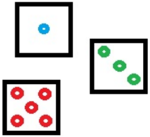

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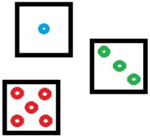

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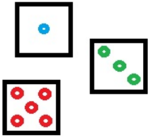

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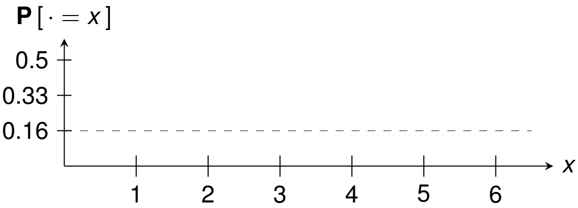

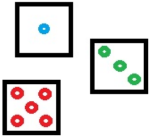

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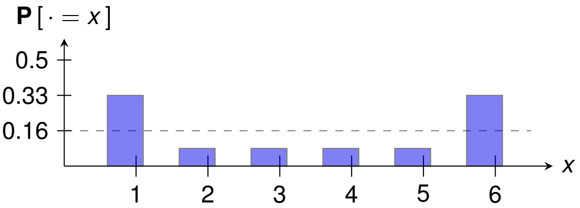

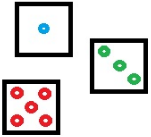

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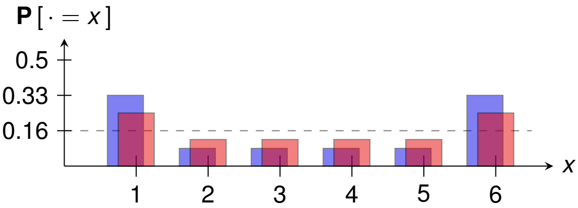

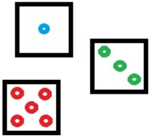

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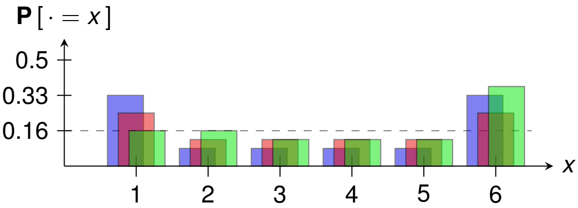

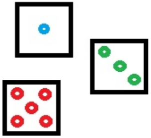

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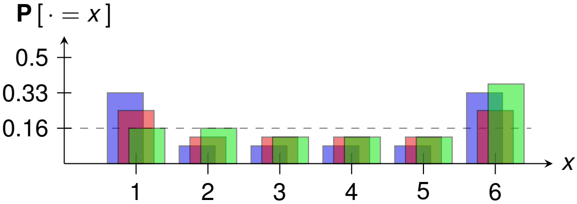

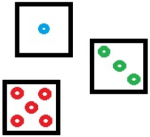

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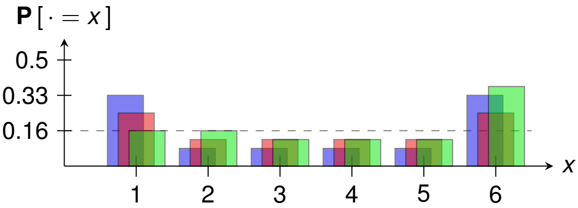

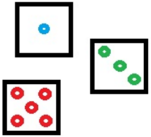

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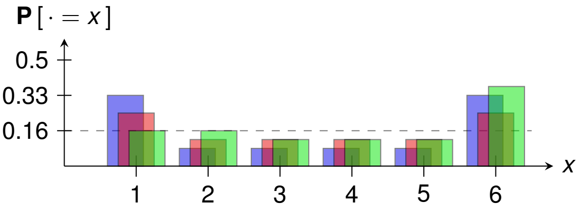

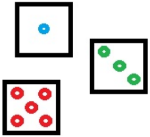

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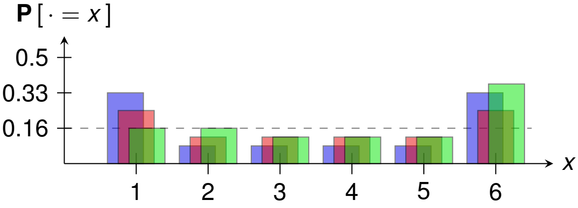

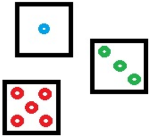

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$$\|D - A\|_{tv} = \frac{1}{2}\left(2\left|\frac{1}{6} - \frac{1}{3}\right| + 4\left|\frac{1}{6} - \frac{1}{12}\right|\right) = \frac{1}{3}$$

$$\|D - B\|_{tv} = \frac{1}{2}\left(2\left|\frac{1}{6} - \frac{1}{4}\right| + 4\left|\frac{1}{6} - \frac{1}{8}\right|\right) = \frac{1}{6}$$

$$\|D - C\|_{tv} = \frac{1}{2}\left(3\left|\frac{1}{6} - \frac{1}{8}\right| + \left|\frac{1}{6} - \frac{9}{24}\right|\right) = \frac{1}{6}.$$

Thus

$$\|D - B\|_{tv} = \|D - C\|_{tv} \qquad \text{and} \qquad \|D - B\|_{tv}, \|D - C\|_{tv} < \|D - A\|_{tv}.$$

So A is the least “fair”, however B and C are equally “fair” (in TV distance).

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We will see a similar result later after introducing spectral techniques (Lecture 12)!

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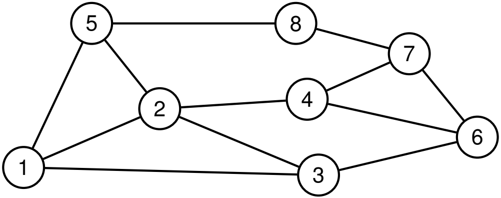
See final slides for some comments on why we choose $1/4$.

Outline

The Independent Set Problem



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Given an undirected graph $G = (V, E)$, an independent set (IS) is a subset $S \subseteq V$ such that there are no two $u, v \in S$ with $\{u, v\} \in E(G)$.

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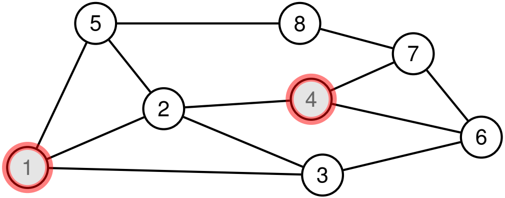

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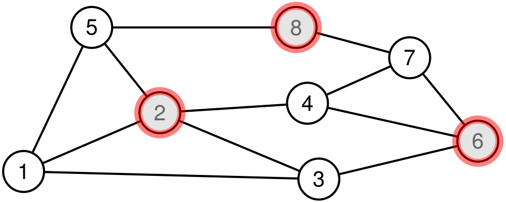

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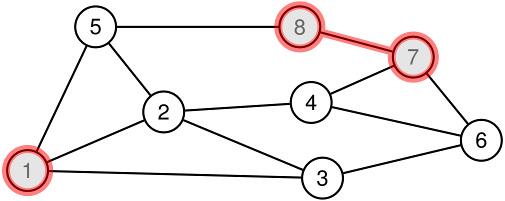

$S = \{1, 7, 8\}$ is **not** an independent set ×

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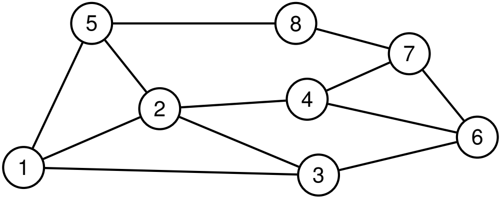

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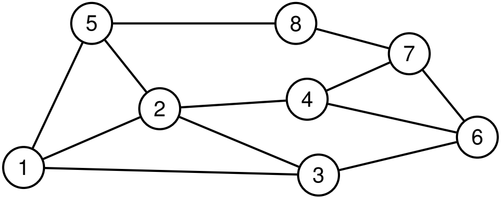

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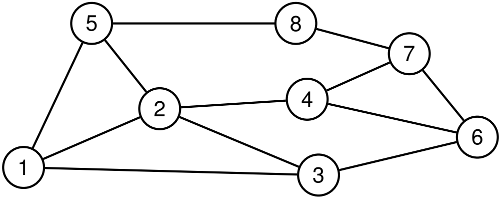

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- Counting the number of independent sets in G is "even harder", it is #P-complete
- Goal: find a randomised approximation algorithm for counting the number of independent sets

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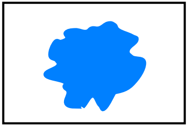
$\Omega = 2^V$



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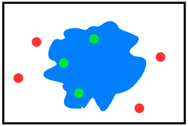

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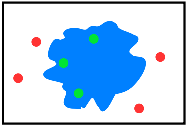

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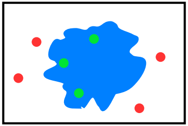

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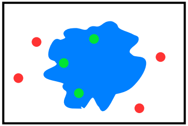

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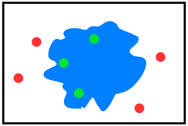

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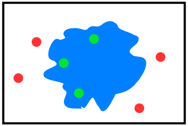

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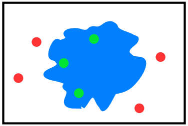

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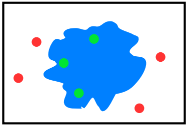

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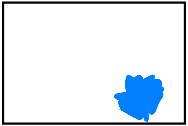

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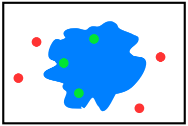

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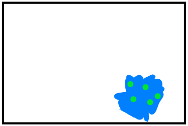

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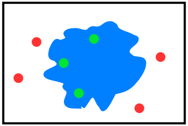

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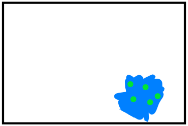

How can we set up a Markov Chain to sample from the set of all IS?

Markov Chain for Sampling Independent Sets

INDEPENDENTSETSAMPLER

Let X_0 be an arbitrary independent set in G
For $t = 0, 1, 2, \ldots$:
    Pick a vertex $v \in V(G)$ uniformly at random
    If $v \in X_t$ **then** $X_{t+1} \leftarrow X_t \setminus \{v\}$
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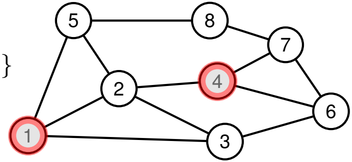
$X_0 = \{1, 4\}$



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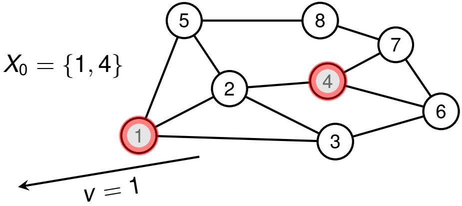

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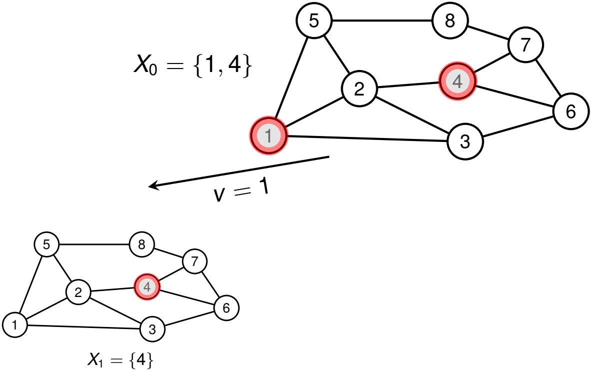

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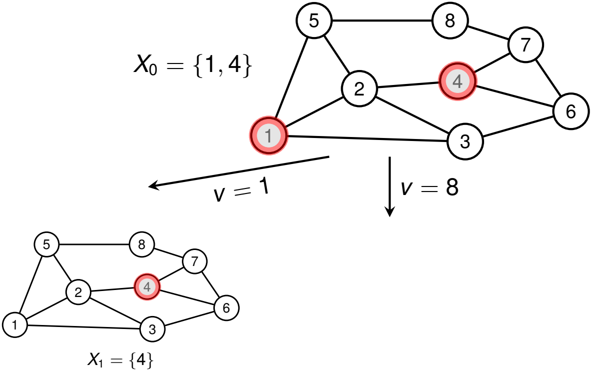

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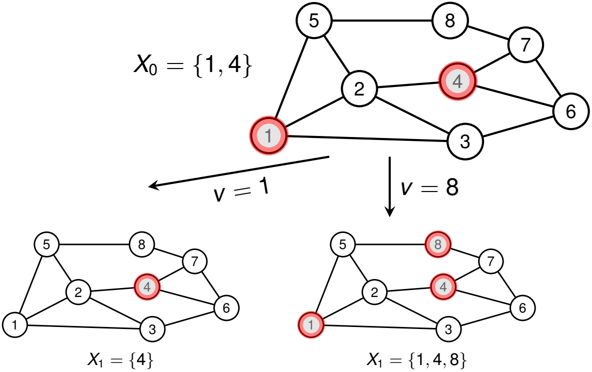

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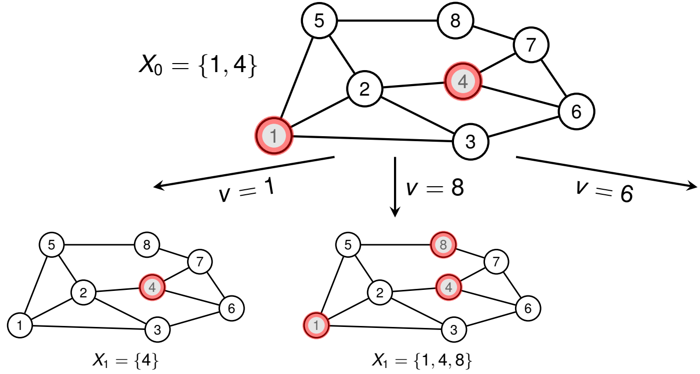

Markov Chain for Sampling Independent Sets

INDEPENDENTSETSAMPLER

Let X_0 be an arbitrary independent set in G
For $t = 0, 1, 2, \dots$:
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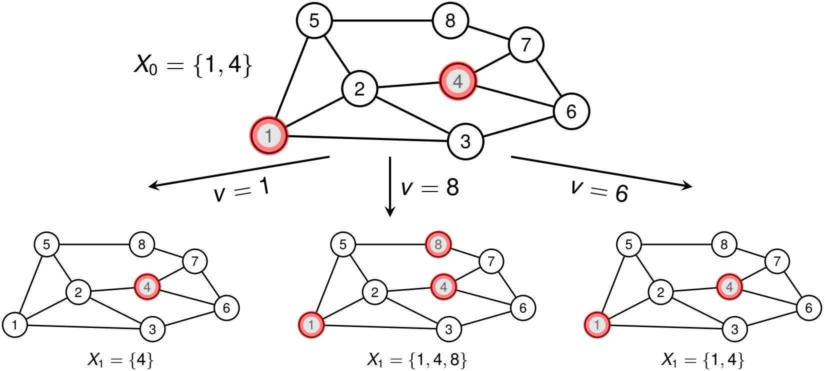

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Key Question: What is the mixing time of this Markov Chain?

Markov Chain for Sampling Independent Sets

IndependentSetSampler

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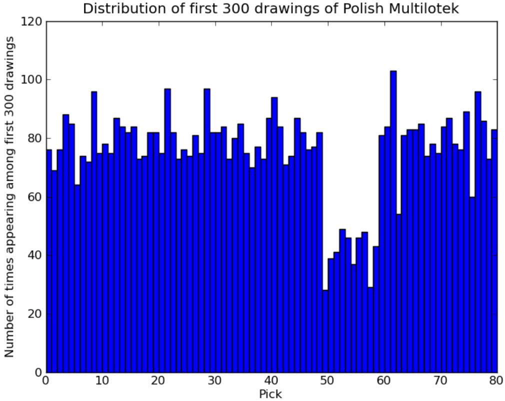
Key Question: What is the mixing time of this Markov Chain?

This is a very deep question and goes beyond the scope of this course. Many positive and negative results are known here, and they often depend on the density of the graph G.

Outline

Experiment Gone Wrong...



Thanks to Krzysztof Onak (pointer) and Eric Price (graph)

Source: Slides by Ronitt Rubinfeld

What is Card Shuffling?

Source: wikipedia

How long does it take to shuffle a deck of 52 cards?

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Persi Diaconis (Professor of Statistics and former Magician)

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One of the leading experts in the field who has related card shuffling to many other mathematical problems.

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Here we will focus on one shuffling scheme which is easy to analyse.

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Here we will focus on one shuffling scheme which is easy to analyse.

How long does it take to shuffle a deck of 52 cards?

How quickly do we converge to the uniform distribution over all $n!$ permutations?

One of the leading experts in the field who has related card shuffling to many other mathematical problems.

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Source: `www.soundcloud.com`

The Card Shuffling Markov Chain

TOPTORANDOMSHUFFLE (Input: A pile of n cards)

For $t = 1, 2, \ldots$
    Pick $i \in \{1, 2, \ldots, n\}$ uniformly at random
    Take the top card and insert it behind the i-th card

The Card Shuffling Markov Chain

TopToRandomShuffle (Input: A pile of n cards)

1: **For** $t = 1, 2, \ldots$
2: Pick $i \in \{1, 2, \ldots, n\}$ uniformly at random
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This is a slightly informal definition, so let us look at a small example...

The Card Shuffling Markov Chain

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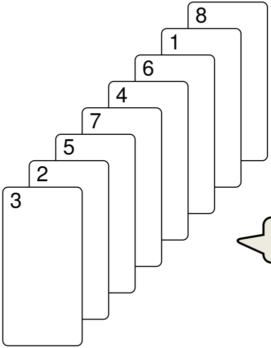
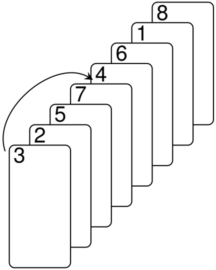


We will focus on this “small” set of cards ($n = 8$)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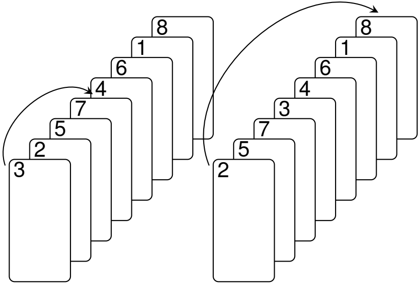
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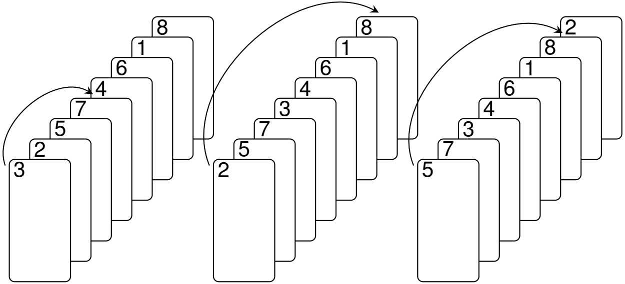
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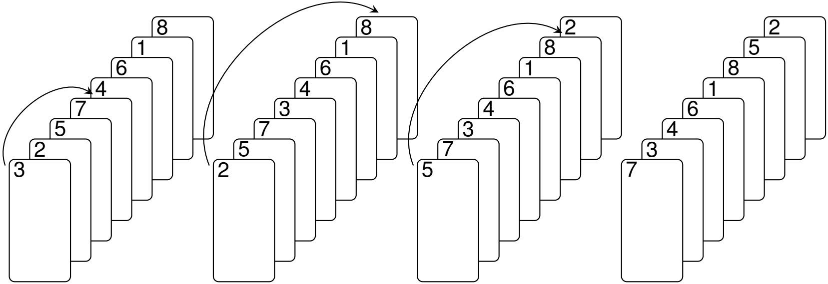
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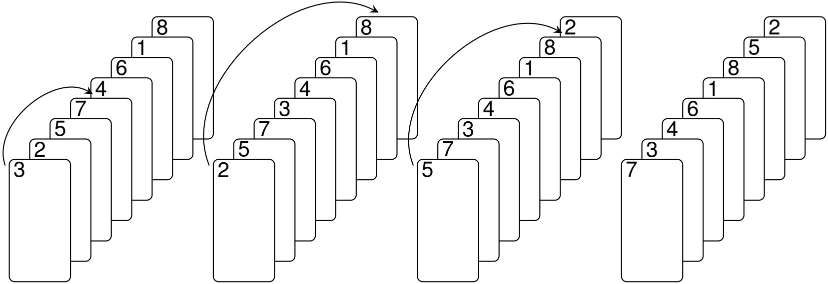
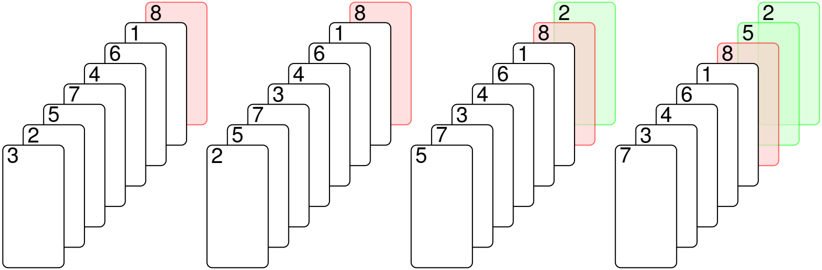


Even if we know which set of cards come after 8, every permutation is equally likely!



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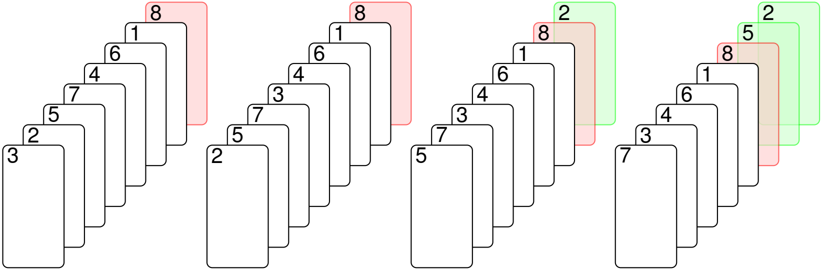

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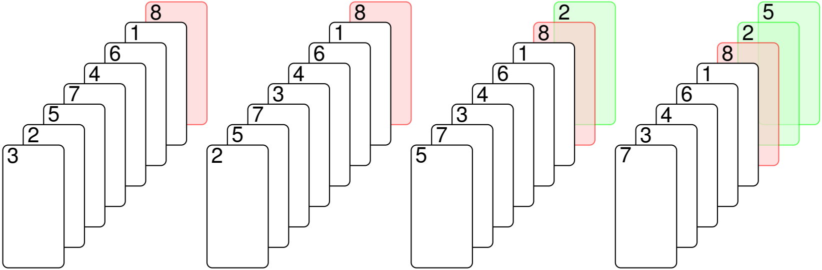

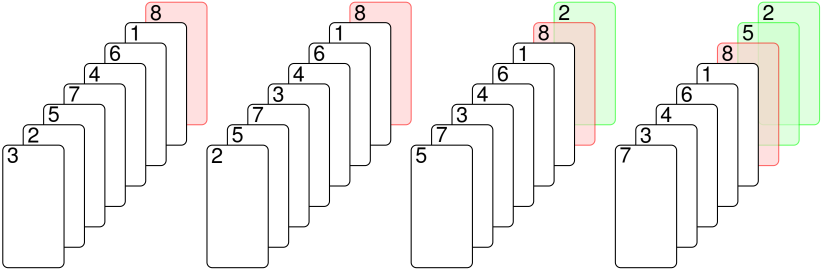

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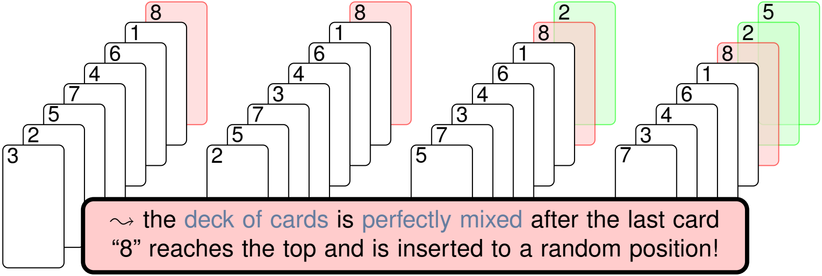
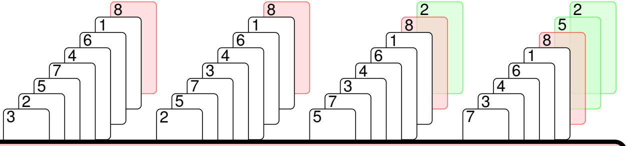


$\rightsquigarrow$ the deck of cards is perfectly mixed after the last card "8" reaches the top and is inserted to a random position!

Analysing the Mixing Time (Intuition)



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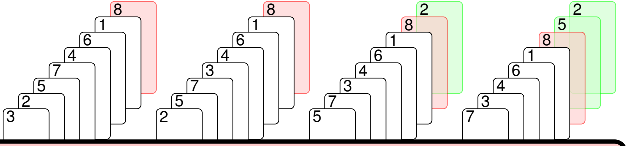

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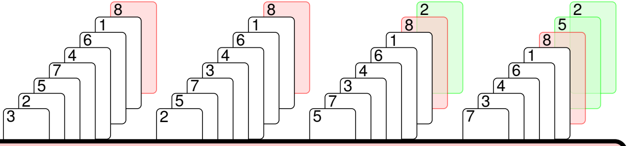

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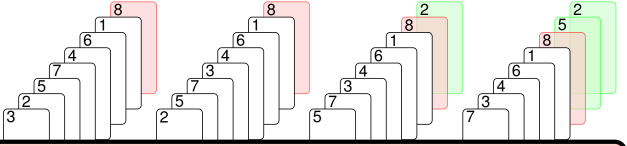

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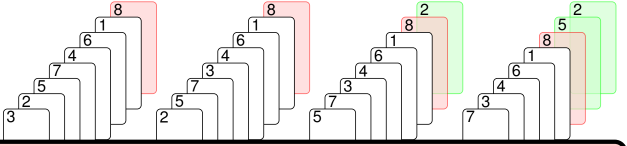

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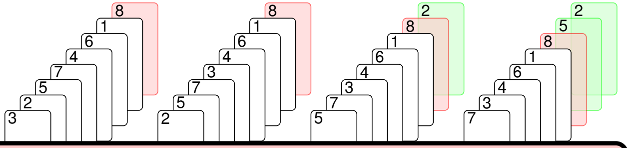

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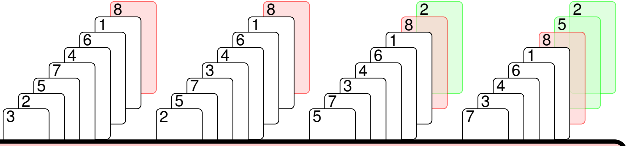

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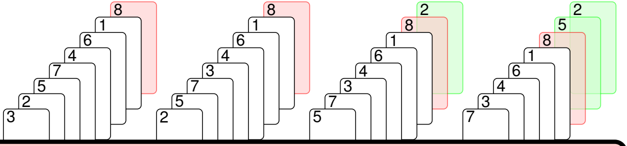

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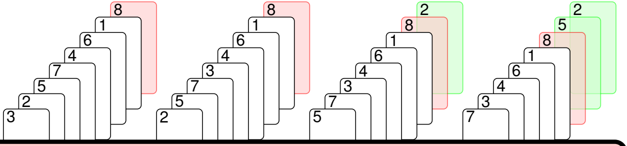

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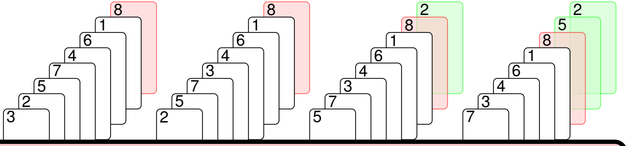

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Using the so-called coupling method, one could prove $t_{mix} \leq n \log n$.

Riffle Shuffle (non-examinable)

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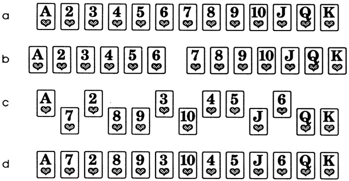

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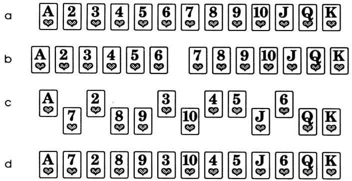

t	1	2	3	4	5	6	7	8	9	10
$\|P^t - \pi\|_{tv}$	1.000	1.000	1.000	1.000	0.924	0.614	0.334	0.167	0.085	0.043

Figure: Total Variation Distance for t riffle shuffles of 52 cards.

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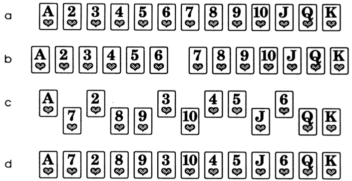

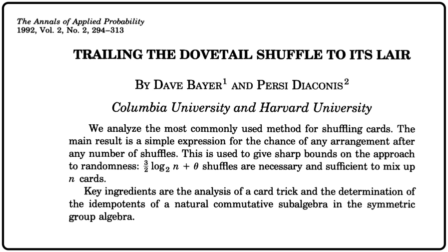

The Annals of Applied Probability
1992, Vol. 2, No. 2, 294–313

TRAILING THE DOVETAIL SHUFFLE TO ITS LAIR

By Dave Bayer[1] and Persi Diaconis[2]

Columbia University and Harvard University

We analyze the most commonly used method for shuffling cards. The main result is a simple expression for the chance of any arrangement after any number of shuffles. This is used to give sharp bounds on the approach to randomness: $\frac{3}{2}\log_2 n + \theta$ shuffles are necessary and sufficient to mix up n cards.

Key ingredients are the analysis of a card trick and the determination of the idempotents of a natural commutative subalgebra in the symmetric group algebra.

t	1	2	3	4	5	6	7	8	9	10
$\|P^t - \pi\|_{tv}$	1.000	1.000	1.000	1.000	0.924	0.614	0.334	0.167	0.085	0.043

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Outline

Further Remarks on the Mixing Time (non-examin.)

- One can prove $\max_x \left\| P_x^t - \pi \right\|_{tv}$ is non-increasing in t (this means if the chain is "ϵ-mixed" at step t, then this also holds in future steps) *[Mitzenmacher, Upfal, 12.3]*

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- We chose $t_{mix} := \tau(1/4)$, but other choices of ϵ are perfectly fine too (e.g, $t_{mix} := \tau(1/e)$ is often used); in fact, any constant $\epsilon \in (0, 1/2)$ is possible.

Further Remarks on the Mixing Time (non-examin.)

- One can prove $\max_x \left\| P_x^t - \pi \right\|_{tv}$ is non-increasing in t (this means if the chain is "ϵ-mixed" at step t, then this also holds in future steps) *[Mitzenmacher, Upfal, 12.3]*
- We chose $t_{mix} := \tau(1/4)$, but other choices of ϵ are perfectly fine too (e.g, $t_{mix} := \tau(1/e)$ is often used); in fact, any constant $\epsilon \in (0, 1/2)$ is possible.

Remark: This freedom on how to pick ϵ relies on the sub-multiplicative property of a (version) of the variation distance. First, let

$$d(t) := \max_x \left\| P_x^t - \pi \right\|_{tv}$$

be the variation distance after t steps when starting from the worst state. Further, define

$$\overline{d}(t) := \max_{\mu,\nu} \left\| P_\mu^t - P_\nu^t \right\|_{tv}.$$

These quantities are related by the following double inequality

$$d(t) \leq \overline{d}(t) \leq 2d(t).$$

Further, $\overline{d}(t)$ is sub-multiplicative, that is for any $s, t \geq 1$,

$$\overline{d}(s+t) \leq \overline{d}(s) \cdot \overline{d}(t).$$

Hence for any fixed $0 < \epsilon < \delta < 1/2$ it follows from the above that

$$\tau(\epsilon) \leq \left\lceil \frac{\ln \epsilon}{\ln(2\delta)} \right\rceil \tau(\delta).$$

In particular, for any $\epsilon < 1/4$

$$\tau(\epsilon) \leq \left\lceil \log_2 \epsilon^{-1} \right\rceil \tau(1/4).$$

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Hence smaller constants $\epsilon < 1/4$ only increase the mixing time by some constant factor.