# 1997 Paper 6 Question 10

## Logic and Proof

State the rules on sequents $\Gamma \Rightarrow \Delta$ involving the universal quantifier in first-order predicate calculus. [2 marks]

Give examples to illustrate the need for the side conditions on variable occurrences. [4 marks]

One of the sequent rules for the universal quantifier makes use of the notion of substituting a term for a variable in a formula. Give an example to show what goes wrong if a free variable in the term being substituted becomes bound or "captured" after substitution. [3 marks]

Let the notation $A\langle t/x \rangle$ denote the result of substituting $t$ for *some* occurrence of $x$ in $A$ if no variable in $t$ becomes bound after substitution. Assume the usual first-order sequent calculus rules (including *cut*), together with all sequents of the form

$$\Gamma \;\Rightarrow\; t = t,\; \Delta$$

and

$$\Gamma,\; t_1 = t_2,\; A\langle t_1/x \rangle \;\Rightarrow\; A\langle t_2/x \rangle,\; \Delta$$

where $t$, $t_1$ and $t_2$ range over arbitrary terms, $x$ is a variable and the substitutions with $t_1$ and $t_2$ are for the same occurrence of $x$.

Give an informal argument that these two rules for "=" are sound principles for reasoning about equality. [2 marks]

Prove that:

(*a*) $\Rightarrow \forall x\, \forall y\, ((x = y) \to (y = x))$ [3 marks]

(*b*) $\Rightarrow \forall x\, \forall y\, \forall z\, ((x = y) \;\wedge\; (y = z) \to (x = z))$ [3 marks]

(*c*) $\Rightarrow \forall y\, ((\exists x\, (x = y) \;\wedge\; P(x)) \to\; P(y))$ [3 marks]