2000 Paper 8 Question 10

Numerical Analysis II

A Riemann integral over $[a, b]$ is defined by

$$\int_a^b f(x)\,dx = \lim_{\substack{n\to\infty \\ \Delta\xi\to 0}} \sum_{i=1}^{n} (\xi_i - \xi_{i-1})\, f(x_i)$$

Explain the terms *Riemann sum* and *mesh norm.* [4 marks]

With respect to an integral over $[-1, 1]$ which of the following are *not* Riemann sums? Give explanations.

(a) $0.2f(-0.9) + 0.8f(-0.1) + 0.8f(+0.1) + 0.2f(+0.9)$

(b) $0.8f(-0.9) + 0.2f(-0.1) + 0.2f(+0.1) + 0.8f(+0.9)$

(c) $0.7f(-0.6) + 0.3f(-0.4) + 0.3f(+0.4) + 0.7f(+0.6)$

(d) $0.5f(-0.7) + 0.8f(0) + 0.5f(+0.7)$

(e) $0.3f(-0.7) + 1.0f(+0.1) + 0.7f(+0.7)$

[5 marks]

Suppose $\mathbf{R}$ is a rule that integrates constants exactly over $[-1, 1]$, and $f(x)$ is bounded and Riemann-integrable over $[a, b]$. Write down a formula for the composite rule $(n \times \mathbf{R})f$ and prove that

$$\lim_{n\to\infty} (n \times \mathbf{R})f = \int_a^b f(x)\,dx$$ [6 marks]

Which of the examples (a) to (e) converge in composite form? [2 marks]

Does the rule

$$-0.5f(-1) + 1.5f(-0.4) + 1.5f(+0.4) - 0.5f(+1)$$

converge in composite form? Comment on its suitability for this purpose.
[3 marks]