2000 Paper 9 Question 10

Numerical Analysis II

Explain the terms (a) *positive definite*, (b) *positive semi-definite* for a symmetric matrix $\mathbf{A}$. If a square matrix $\mathbf{B}$ is non-singular, which of the properties (a) or (b) most accurately describes $\mathbf{B}^T\mathbf{B}$? What if $\mathbf{B}$ is singular? [4 marks]

State *Schwarz's inequality* for the product $\mathbf{AB}$. In what way is this modified for the product $\mathbf{Ax}$, where $\mathbf{x}$ is a vector? What are the *singular values* of $\mathbf{A}$, and how are they related to the l_2 norm of $\mathbf{A}$? In the *singular value decomposition* $\mathbf{A} = \mathbf{UWV}^T$, what is $\mathbf{W}$? [5 marks]

Let $\hat{\mathbf{x}}$ be an approximate solution of $\mathbf{Ax} = \mathbf{b}$, and write $\mathbf{r} = \mathbf{b} - \mathbf{A}\hat{\mathbf{x}}$, $\mathbf{e} = \mathbf{x} - \hat{\mathbf{x}}$. Find an expression which is an upper bound for the relative error $||\mathbf{e}||/||\mathbf{x}||$ in terms of computable quantities. Explain how this result may be interpreted if the l_2 norm is used. [8 marks]

Suppose $\mathbf{A}$ is a 5×5 matrix and $\mathbf{Ax} = \mathbf{b}$ is to be solved by singular value decomposition. If *machine epsilon* $\simeq 10^{-15}$ and the singular values of $\mathbf{A}$ are $1, 10^{-6}, 10^{-10}, 10^{-17}, 0$ write down the generalised inverse $\mathbf{W}^+$ that you would use. [3 marks]