2002 Paper 10 Question 8

Continuous Mathematics

Consider the trigonometric series

$$\frac{a_0}{2} + \sum_{r=1}^{\infty} (a_r \cos rx + b_r \sin rx)$$

where $a_0, a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ are constants and suppose that $f(x)$ is a periodic function of x with period 2π.

(a) State expressions for the constants a_0, a_r, b_r $(r = 1, 2, \ldots)$ so that the trigonometric series forms the *Fourier series* of $f(x)$ over the interval $-\pi < x \leqslant \pi$. Such expressions are then known as the *Fourier coefficients* of $f(x)$. [4 marks]

(b) State the *Dirichlet conditions* on the function $f(x)$ for it to be represented by its Fourier series at all points in the interval $-\pi < x \leqslant \pi$ at which the function $f(x)$ is continuous. [2 marks]

(c) Determine simplified expressions for the Fourier coefficients when the function $f(x)$ is an even function of x. [3 marks]

(d) Consider the function $f(x)$ which is periodic with period 2π and is defined by $f(x) = x^2$ in the interval $-\pi < x \leqslant \pi$. Does the function $f(x)$ satisfy the Dirichlet conditions? Briefly justify your answer. [2 marks]

(e) Determine the Fourier series for this function $f(x)$. [6 marks]

(f) By substituting a suitable value for x in the Fourier series show that

$$\frac{\pi^2}{12} = \sum_{r=1}^{\infty} \frac{(-1)^{r+1}}{r^2}.$$

[3 marks]