2002 Paper 1 Question 8

Discrete Mathematics

Let Ω be a set. Write $\mathcal{P}(\Omega)$ for its powerset. Recall the definition of the intersection of $\mathcal{B} \subseteq \mathcal{P}(\Omega)$:

$$\bigcap_{B \in \mathcal{B}} B = \{x \in \Omega \mid \forall B \in \mathcal{B}.\ x \in B\}\ .$$

(*a*) Let $\mathcal{B} \subseteq \mathcal{P}(\Omega)$ and $\mathcal{C} \subseteq \mathcal{P}(\Omega)$.

(*i*) Prove that

$$(\bigcap_{B \in \mathcal{B}} B) \cup (\bigcap_{C \in \mathcal{C}} C) \ \subseteq \ \bigcap_{(B,C) \in \mathcal{B} \times \mathcal{C}} (B \cup C)\ .$$

[3 marks]

(*ii*) Prove that

$$\bigcap_{(B,C) \in \mathcal{B} \times \mathcal{C}} (B \cup C) \ \subseteq \ (\bigcap_{B \in \mathcal{B}} B) \cup (\bigcap_{C \in \mathcal{C}} C)\ .$$

[6 marks]

(*b*) Let $\mathcal{A} \subseteq \mathcal{P}(\Omega)$. Suppose that $\mathcal{A}$ is *intersection-closed* in the sense that

$$\text{if } \mathcal{B} \subseteq \mathcal{A}, \text{ then } \bigcap_{B \in \mathcal{B}} B \in \mathcal{A}\ .$$

Define

$$\mathcal{R} = \{(X, y) \in \mathcal{P}(\Omega) \times \Omega \mid \forall A \in \mathcal{A}.\ X \subseteq A \Rightarrow y \in A\}\ .$$

Let $C \subseteq \Omega$. Say C is $\mathcal{R}$-*closed* iff

$$\forall (X, y) \in \mathcal{R}.\ X \subseteq C \Rightarrow y \in C\ .$$

You are asked to show that the members of $\mathcal{A}$ are precisely the $\mathcal{R}$-closed subsets of Ω, in the following two stages:

(*i*) Show

$$\text{if } C \in \mathcal{A}, \text{ then } C \text{ is } \mathcal{R}\text{-closed}\ .$$

[2 marks]

(*ii*) Show

$$\text{if } C \text{ is } \mathcal{R}\text{-closed, then } C \in \mathcal{A}\ .$$

[Hint: Consider the set $\mathcal{B} = \{A \in \mathcal{A} \mid C \subseteq A\}$.] [9 marks]