2002 Paper 2 Question 4

Probability

Following a hardware practical class, an undergraduate comes across a tangle of n (single-strand) wires. After finding the $2n$ ends, he idly joins these ends together in pairs until there are no loose ends left. [At each joining, the first end is selected equiprobably from the unjoined ends and joined to a second end which is likewise selected equiprobably from the remainder.]

Another undergraduate later comes across the handiwork of the first and decides to count the number of loops in the tangle. She reasons that there could be anything from one long loop to n separate loops.

Let X_n (where $n \geqslant 1$) be a random variable whose value r (where $1 \leqslant r \leqslant n$) is the number of loops that result when the $2n$ ends of n wires are joined in pairs. Note the special case, $\mathrm{P}(X_1 = 1) = 1$. Write down a difference equation which expresses $\mathrm{P}(X_n = r)$ in terms of $\mathrm{P}(X_{n-1} = r - 1)$ and $\mathrm{P}(X_{n-1} = r)$ given $1 < r < n$ and (in consequence) $n \geqslant 3$. [8 marks]

Write down the difference equations for the special cases $\mathrm{P}(X_n = 1)$ and $\mathrm{P}(X_n = n)$. These equations should hold for $n \geqslant 2$. [4 marks]

Hence or otherwise, tabulate the distributions of the random variables X_1, X_2 and X_3 expressing all values as fractions. [5 marks]

What is the Expectation $\mathrm{E}(X_3)$, again expressed as a fraction? [3 marks]