2004 Paper 1 Question 7

Discrete Mathematics

Recall the Fibonacci numbers defined by:

- $f_0 = 0$
- $f_1 = 1$
- $f_n = f_{n-1} + f_{n-2}$ for $n > 1$

Using induction on n, or otherwise, show that $f_{m+n} = f_{m-1}f_n + f_m f_{n+1}$ for $m > 0$. [4 marks]

Deduce that $\forall m, n > 0 \,.\, m|n \Rightarrow f_m|f_n$. [4 marks]

Deduce further that $\forall n > 4 \,.\, f_n$ prime $\Rightarrow n$ prime. [2 marks]

Given $n \in \mathbb{N}$, let $g_i = f_i \bmod n$, and consider the pairs (g_1, g_2), $(g_2, g_3), \ldots, (g_i, g_{i+1}), \ldots$. Show that there must be a repetition in the first $n^2 + 1$ pairs. Let $r < s$ be the least values with $(g_r, g_{r+1}) = (g_s, g_{s+1})$. Show that $g_{r-1} = g_{s-1}$, and deduce that $r = 1$. Calculate g_1 and g_2, and deduce that $g_{s-1} = 0$. Hence show that one of the first n^2 Fibonacci numbers is divisible by n. [10 marks]